Applied Mathematics III

Time : 3 Hrs.] Prelim Question Paper Solution [Marks : 80

Q.1(a) Find Laplace Transformation of $\frac{\sin(3t)}{t}$ **[5]**

(A) We know L {sin3t} $= \frac{3}{s^2+3^2}$

$$\therefore\ L\left[\frac{\sin(3t)}{t}\right] = \int_{s=3}^{\infty}\frac{3}{s^2+3^2}ds = \frac{3}{3}\tan^{-1}(s/3)\Big|_{s=3}^{\infty}$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s/3) = \frac{\pi}{2} - \tan^{-1}(s/3)$$

$$\therefore\ L\left(\frac{\sin(3t)}{t}\right) = \cot^{-1}(s/3)$$

Q.1(b) Prove that f(z) = cosh z is analytic and find its derivative. **[5]**

(A) f(z) = cosh(z) = cosh(x+iy) = coshx cosy + i sinhx siny = u + iv

$\therefore$ u = coshx cosy, v = sinhx siny

u_x = sinhx cosy, u_y = -coshx siny, v_x = coshx siny, v_y = sinhx cosy

$\therefore$ $u_x = v_y$ and $u_y = -v_x$

$\therefore$ u and V satisfies C.R. equations

$\therefore$ f(z) = u + iv is analytic

Now $ux|_{(z,0)}$ = sinhz $vx|_{(z,0)}$ = 0

f'(z) $= u_x + iv_x$

= sinhx cosy + i sinhx siny

put x = z, y = 0

f'(z) = sinhz

$\therefore$ $f'(z)|_{(z,0)} = ux|_{(z,0)} + ivx|_{(z,0)}$

= sinhz + i0 = sinhz

Q.1(c) Find Fourier series for f(x) = 16 − x^2 over (–4, +4) **[5]**

(A) $\because$ f(x) is even function in (-4, 4) $\therefore$ its Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} an\cos\left(\frac{n\pi x}{4}\right) \quad \text{... (1)}$$

where $a_0 = \frac{2}{4}\int_{x=0}^{4} f(x)\,dx = \frac{1}{2}\int_{x=0}^{4}(16-x^2)dx = \frac{1}{2}\left[16x - \frac{x^3}{3}\right]_0^4 = \frac{1}{2}\left[64 - \frac{64}{3}\right]$

$$\frac{32}{3}\times 2 = \frac{64}{3} \qquad \ldots (2)$$

$$a_n = \frac{2}{4}\int_{x=0}^{4} f(x)\cos\left(\frac{n\pi x}{4}\right)dx = \frac{1}{2}\int_{x=0}^{4}\left(16-x^2\right)\cos\left(\frac{n\pi x}{4}\right)dx$$

$$a_n = \frac{1}{2}\left[\left(16-x^2\right)\frac{\sin\left(n\pi x/4\right)}{n\pi/4} - \left(-2x\right)\left(\frac{-\cos\left(n\pi x/4\right)}{\left(n\pi/4\right)^2}\right) + (-2)\left(\frac{-\sin\left(n\pi x/4\right)}{\left(n\pi/4\right)^3}\right)\right]_{x=0}^{4}$$

$$a_n = \frac{1}{2}\left[\frac{4}{n\pi}\left(16-x^2\right)\sin\left(\frac{n\pi x}{4}\right) - \frac{16}{n^2\pi^2}\times 2\ x\cos\left(\frac{n\pi x}{4}\right) + \frac{64}{n^3\pi^3}\times 2\sin\left(\frac{n\pi x}{4}\right)\right]_{x=0}^{4}$$

$$a_n = \frac{1}{2}\left[\frac{4}{n\pi}(0-0) - \frac{32}{n^2\pi^2}\left[4\left(-1\right)^n - 0\right] + \frac{128}{n^3\pi^3}\left[0-0\right]\right] = \frac{-1}{2}\frac{32}{n^2\pi^2}4\left(-1\right)^n$$

$$\therefore a_n = \frac{-64}{n^2\pi^2}\left(-1\right)^n \qquad \ldots (3)$$

From (1), (2), (3)

$$f(x) = \frac{1}{2}\left(\frac{64}{3}\right) + \sum_{n=1}^{\infty}\frac{-64}{n^2\pi^2}\left(-1\right)^n\cos\left(\frac{n\pi x}{4}\right)$$

$$\therefore \quad f(x) = \frac{32}{2} - \frac{64}{\pi^2}\sum_{n=1}^{\infty}\frac{\left(-1\right)^n}{n^2}\cos\left(\frac{n\pi x}{4}\right)$$

Q.1(d) Find Z{f(k)·g(k)} if f(k) = $\frac{1}{3^k}$ & g(k) = $\frac{1}{7^k}$ **[5]**

(A) Let $f(k) = \frac{1}{3^k}$ and $g(k) = \frac{1}{7^k}$

$$\therefore \ \bar{F}(z) = z\left[f(k)\right] = \sum_{k=0}^{\infty} f(k)\, z^{-k} = \sum_{k=0}^{\infty}\frac{1}{3^k}\cdot\frac{1}{z^k}$$

$$\therefore \ \bar{F}(z) = \sum_{k=0}^{\infty}\left(\frac{1}{3z}\right)^0 = 1 + \frac{1}{3z} + \left(\frac{1}{3z}\right)^2 + \left(\frac{1}{3z}\right)^3 \ldots = \frac{1}{1-\frac{1}{3z}} \qquad \left|\frac{1}{3z}\right| < 1$$

$$\bar{F}(z) = \frac{3z}{3z-1}, \text{ For } |z| > \frac{1}{3}$$

$$\text{Similarly } \bar{G}(z) = \sum_{k=0}^{\infty} z(k) = \sum_{k=0}^{\infty}\left(\frac{1}{7^k}\right) = \frac{7z}{7z-1} \qquad |z| > \frac{1}{7}$$

∴ By convolution Theorem

$$z\{f(k)\cdot g(k)\} = \bar{F}(z)\cdot\bar{G}(z) = \frac{3z}{3z-1}\cdot\frac{7z}{7z-1} = \frac{21z^2}{21z^2-10z+1} \quad \text{for } |z| > \frac{1}{3} > \frac{1}{7}$$

Q.2(a) Show that the vector field $\bar{F} = (3x^2 y)i + (x^3 - 2yz^2)j + (3z^2 - 2y^2 z)k$ is conservative and find $\phi(x, y, z)$ such that $\bar{F} = \Delta\phi$. Also evaluate the line integral $\int \bar{F} \cdot d\bar{r}$ from (2, 1, 1) to (2, 0, 1) **[6]**

(A)
$$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y & x^3 - 2yz^2 & 3z^2 - 2y^2z \end{vmatrix} = \begin{array}{l} i\left(-4yz - (-4yz)\right) \\ -j\left(0-0\right) \\ +k\left(3x^2 - 3x^2\right) \end{array}$$

$\therefore\ \nabla \times \bar{F} = oi + oj + ok = \bar{0}$

$\therefore\ \bar{F}$ is irrotational

$\therefore$ There exist scalar point function ϕ such that

$\bar{F} = \nabla\phi$

$\because\ \phi = \phi(x, y, z)$

$$\therefore\ d\phi = \nabla\phi \cdot d\bar{z} = \bar{F} \cdot d\bar{z} = (3x^2y)dx + (x^3 - 2yz^2)dy + (3z^2 - 2y^2z)dz$$

$$\therefore\ \phi(x, y, z) = \int (3x^2y)dx + (x^3 - 2yz^2)dy + (3z^2 - 2y^2z)dz$$

$$\therefore\ \phi(x, y, z) = \frac{3x^3}{3}y + \left(\frac{-2y^2}{2}z^2\right) + z^3 + c$$

$\therefore\ \phi(x, y, z) = x^3y - y^2z^2 + z^3 + c$ be the required scalar point function.

$$\therefore \int_{(2,1,1)}^{(2,0,1)} \bar{F} \cdot d\bar{z} = \int_{(2,1,1)}^{(2,0,1)} \left[(3x^2y)dx + (x^3 - 2yz^2)dy + (3z^2 - 2y^2z)dz\right]$$

$$= x^3y - y^2z^2 + z^3 + c\Big|_{(2,1,1)}^{(2,0,1)} = 1 - [8 - 1 + 1] = -7$$

$$\therefore \int_{(2,1,1)}^{(2,0,1)} \bar{F} \cdot d\bar{z} = -7$$

Q.2(b) Find the Fourier series for $f(x) = \frac{x - \pi}{4}$; $0 \le x \le 2\pi$. Hence prove that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$ **[6]**

(A) We know fourier series for the function in $[0, 2\pi]$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_v \sin(nx) \quad \ldots (1) \text{ where}$$

$$a_0 = \frac{1}{\pi}\int_{x=0}^{2\pi} f(x)dx,\ a_n = \frac{1}{\pi}\int_{x=0}^{2\pi} f(x)\cos(nx)dx,\ b_n = \frac{1}{\pi}\int_{x=0}^{2\pi} f(x)\sin(nx)dx$$

$$a_0 = \frac{1}{\pi}\int_{x=0}^{2\pi} \frac{1}{4}(x - \pi)dx = \frac{1}{4\pi}\frac{(x-\pi)^2}{2}\Bigg|_0^{2\pi} = \frac{1}{8\pi}\left[\pi^2 - (-\pi)^2\right] = 0 \quad \ldots (2)$$

$$a_n = \frac{1}{\pi}\int_{x=0}^{2\pi}\left(\frac{x-\pi}{4}\right)\cos(nx)\,dx = \frac{1}{4\pi}\left[(x-\pi)\left[\frac{\sin(nx)}{n}\right]-1\left[\frac{-\cos(nx)}{n^2}\right]\right]_{x=0}^{2\pi}$$

$$\therefore \quad a_n = \frac{1}{4\pi}\left[\frac{1}{n}(x-\pi)\sin(nx)+\frac{1}{n^2}\left[\cos(nx)\right]\right]_{x=0}^{2\pi}$$

$$a_n = \frac{1}{4\pi}\left[\frac{1}{n}\left[0-0\right]+\frac{1}{n^2}\left[1-1\right]\right] = 0 \qquad \text{... (3)}$$

$$b_n = \frac{1}{\pi}\int_{x=0}^{2\pi}\left(\frac{x-\pi}{4}\right)\sin(nx)\,dx = \frac{1}{4\pi}\left[(x-\pi)\left(\frac{-\cos(nx)}{n}\right)-1\left(\frac{-\sin(nx)}{n^2}\right)\right]_{x=0}^{2\pi}$$

$$b_n = \frac{1}{4\pi}\left[-\frac{1}{n}(x-\pi)\cos(nx)+\frac{1}{n^2}\sin(nx)\right]_{x=0}^{2\pi}$$

$$= \frac{1}{4\pi}\left[-\frac{1}{n}\left[\pi\cdot 1-(-\pi)\cdot 1\right]+\frac{1}{n^2}(0-0)\right]$$

$$b_n = \frac{1}{4\pi}\left[-\frac{1}{n}(2\pi)\right] = \frac{-1}{4\pi}\cdot\frac{2\pi}{n} = \frac{-1}{2n} \qquad \text{...(4)}$$

From (1), (2), (3) and (4)

$$f(x) = \frac{1}{2}\cdot 0+\sum_{n=1}^{\infty}0\cdot\cos(nx)+\sum_{n=1}^{\infty}\left(\frac{-1}{2n}\right)\sin(nx)$$

$$\therefore\ f(x) = -\frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}$$

$$\therefore\ \frac{1}{4}(x-\pi) = -\frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{\sin(nx)}{n}\right], \qquad 0\le x\le 2\pi$$

Put $x = \pi/2$

$$\frac{1}{4}\left(\frac{\pi}{2}-\pi\right) = \frac{-1}{2}\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{n}$$

$$\therefore\quad \frac{1}{4}\left(\frac{-\pi}{2}\right) = \frac{1}{2}\left[\frac{1}{1}+0-\frac{1}{3}+0+\frac{1}{5}+0-\frac{1}{7}\ldots\right]$$

$$\therefore\quad \frac{\pi}{4} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$$

Q.2(c) Find Inverse Laplace transform of **[8]**

(i) $\frac{s+19}{(s+9)(s^2+4)}$, **(ii)** $\frac{e^{-3s}}{(s^2+10s+29)}$.

(A) (i) $\frac{s+19}{(s+9)(s^2+4)}$

Let $\frac{s+19}{(s+9)(s^2+4)} = \frac{A}{s+9}+\frac{Bs+C}{s^2+4}$ ... (1)

$\therefore$ $A(s^2 + 4) + (Bs + C)(s + 9) = s + 19$ ... (2)

if $s = -9$, $85A = +10$ $\therefore A = \frac{2}{17}$

if $s = 04$ $A + 9C = 19$

$$9C = 19 - 4A = 19-\frac{8}{17} = \frac{315}{17}$$

$$\therefore \quad C = \frac{35}{17}$$

if $s = 1$, $5A + 10(B + C) = 20$

$$B + C = \frac{1}{10}\left[20-5A\right] = \frac{1}{10}\left[20-\frac{10}{17}\right] = \left[2-\frac{1}{17}\right] = \frac{33}{17}$$

$$B = \frac{33}{17}-C = \frac{33}{17}-\frac{35}{17}$$

$$B = \frac{-2}{17}$$

$$\therefore \frac{s+19}{(s+9)(s^2+4)} = \frac{2}{17}\frac{1}{s+9}+\frac{\frac{-2}{17}s+\frac{35}{17}}{s^2+4} = \frac{2}{17}\cdot\frac{1}{s+9}+\frac{1}{17}\frac{\left[35-2s\right]}{s^2+4}$$

$$= \frac{1}{17}\left[2\frac{1}{s+g}+\frac{35}{2}\left(\frac{2}{s^2+2^2}\right)-\frac{2s}{s^2+2^2}\right]$$

$$\therefore L^{-1}\left[\frac{s+19}{(s+9)(s^2+4)}\right] = \frac{1}{17}\left[2L^{-1}\left(\frac{1}{s+9}\right)+\frac{35}{2}L^{-1}\left(\frac{2}{s^2+2^2}\right)-2L^{-1}\left(\frac{s}{s^2+2^2}\right)\right]$$

$$= \frac{1}{17}\left[2e^{-9t}+\frac{35}{2}\sin(2t)-2\cos(2t)\right]$$

(ii) Let $\bar{f}(s) = \frac{1}{s^2+10s+29} = \frac{1}{s^2+10s+25+4} = \frac{1}{(s+5)^2+2^2}$

$$\therefore \quad L^{-1}\left[\bar{f}(s)\right] = L^{-1}\left[\frac{1}{(s+5)^2+2^2}\right] = e^{-5t}L^{-1}\left(\frac{1}{s^2+2^2}\right) = \frac{e^{-5t}}{2}\sin(2t)$$

$$\therefore\ L^{-1}\left[e^{-3s}\frac{1}{s^2+10s+29}\right] = L^{-1}\left[e^{-3s}\,\bar{f}(s)\right] = h(t-a)f(t-a)$$

$$= h(t-3)\cdot\frac{e^{-5(t-3)}}{2}\sin\left[2(t-3)\right]$$

Q.3(a) Find the analytic function f(z) = u + iv **[6]**

if u + v = $\dfrac{2\sin(2x)}{e^{2y}+e^{-2y}-2\cos(2x)}$

(A) $u + (v) = \dfrac{2\sin(2x)}{2\cosh(2y)-2\cos(2x)} = \dfrac{\sin(2x)}{\cosh(2y)-\cos(2x)}$ = (v)

By given f(z) = u + iv $\therefore$ i f(z) = iu − v ... (1)

$\therefore$ (1 + i) f(z) = (u - v) + i(u + v)

$\therefore$ F(z) = U + iV ... (2)

where F(z) = (1 + i) f(z)

u – v = U and u + v = V ... (3)

is also analytic

Now $F'(z) = U_{xei}V_x = V_y + iV_x$

$\therefore F'(z)|_{(z,\,0)} = V_y|_{(z,\,0)} + iV_x|_{(z,\,0)}$ ... (4)

$$V_x = \frac{\left[\cosh(2y)-\cos(2x)\right]2\cos(2x)-\sin(2x)\left[2\sin(2x)\right]}{\left(\cosh 2y-\cos 2x\right)^2}$$

$$V_x|_{(z,\,0)} = \frac{2\left[\cos 2z(1-\cos 2z)-\sin^2(2z)\right]}{(1-\cos 2z)^2} = \frac{2\left[\cos(2z)-\cos^2(2z)-\sin^2(2z)\right]}{(1-\cos 2z)^2}$$

$$= \frac{-2\left[1-\cos(2z)\right]}{(1-\cos 2z)^2} = \frac{-2}{1-\cos 2z} = \frac{-2}{2\sin^2 z} = -\operatorname{cosec}^2 z \quad \text{... (5)}$$

$$V_y = \frac{-\sin(2xy)}{(\cosh 2y-\cos 2xy)}\times 2\sinh(2y)$$

$$\therefore\quad V_y|_{(z,0)} = \frac{-2\sin(2y)0}{(1-\cos 2y)^2} = 0 \quad \text{... (6)}$$

Now from (4), (5) and (6)

$F'(z)|_{(z,\,0)} = 0 + i(-\operatorname{cosec}^2 z)$ = -icosec^2(z)

$\therefore$ F(z) = $\int F'(z)|_{(z,0)}\,dz + c = -i\int \operatorname{cosec}^2 z\,dz + c = i(+\cot z) + c$

$\therefore$ (1 + i) f(z) = +icot z + c

$$\therefore\ f(z) = \frac{1}{1+i} i\cot z + \frac{c}{1+i} = \frac{(1-i)i}{2}\cot z + c' = \frac{(1+i)}{2}\cot z + c'$$

Q.3(b) Find inverse Z transformation of $\dfrac{1}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}$, $\dfrac{1}{5} < |z| < \dfrac{1}{4}$ **[6]**

(A) Let $\dfrac{1}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)} = \dfrac{A}{z-\frac{1}{4}} + \dfrac{B}{z-\frac{1}{5}}$ ...(1)

$$\therefore\ A\left(z-\frac{1}{5}\right) + B\left(z-\frac{1}{4}\right) = 1 \qquad ...(2)$$

if $z = \frac{1}{4}$ $\therefore A\left(\frac{1}{4}-\frac{1}{5}\right) = 1$ $\therefore \frac{A}{20} = 1$ $\therefore A = 20$

if $z = \frac{1}{5}$ $\therefore B\left(\frac{1}{5}-\frac{1}{4}\right) = 1$ $\therefore \frac{-B}{20} = 1$ $\therefore B = -20$

$$\therefore\ \frac{1}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)} = \frac{20}{z-\frac{1}{4}} - \frac{20}{z-\frac{1}{5}} = 20\left[\frac{1}{z-\frac{1}{4}} - \frac{1}{z-\frac{1}{5}}\right]$$

$$\therefore\ \bar{F}(z) = 20\left[\frac{-4}{1-4z} - \frac{1}{z\left(1-\frac{1}{5z}\right)}\right] = -80\left(\frac{1}{1-4z}\right) - \frac{20}{z}\left(\frac{1}{1-\frac{1}{5z}}\right)$$

$$\bar{F}(z) = -80\left[1+(4z)+(4z)^2 \ldots\right] - \frac{20}{z}\left[1+\left(\frac{1}{5z}\right)+\left(\frac{1}{5z}\right)^2 \ldots\right]$$

$$= -80\sum_{k=0}^{\infty} 4^k \cdot z^k - \frac{20}{z}\sum_{k=0}^{\infty}\frac{1}{5^k z^k} = -80\sum_{n=-\infty}^{0} 4^{-n} z^{-n} - 20\sum_{k=0}^{\infty}\frac{1}{5^k z^{k+1}}$$

$$\bar{F}(z) = -80\sum_{k=-\infty}^{0} 4^{-n} z^{-n} - 20\sum_{n=1}^{\infty}\frac{1}{5^{n-1}}\frac{1}{z^n}$$

$$= \sum_{k=-\infty}^{0}\left(-80\times 4^{-k}\right)z^{-k} + \sum_{k=1}^{\infty} -20\left(\frac{1}{5}\right)^{k-1} \cdot z^{-k}$$

$$\therefore\ |\ Z^{-1}\left[\bar{F}(z)\right] = -80\times 4^{-k} \qquad k \le 0$$

$$= -\frac{20}{5^{k-1}} \qquad k \ge 1$$

$$\text{ie}\quad Z^{-1}\left[\frac{1}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}\right] = \begin{cases} -80.4^{-k} & k \le 0 \\ \dfrac{-20}{5^{k-1}} & k \ge 1 \end{cases}$$

Q.3(c) **Solve the differential equation $\frac{d^2y}{dt^2}+4y = f(t)$, with y(0) = 0 & y′(0) = 1 and $f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & 1 < t \end{cases}$** **[8]**

(A) By given $y^{11}(t) + 4y(t) = f(t)$

$$\therefore L[y''(t)] + 4\,Ly(t) = Lf(t) = \int_{t=0}^{\infty} e^{-st} f(t)dt = \int_{t=0}^{1} e^{-st} f(t)dt + \int_{t=1}^{\infty} e^{-st} f(t)dt$$

$$s^2\bar{y}(s) - sy(0) - y'(0) + 4\bar{y}(s) = \int_{t=0}^{1} e^{-st}\cdot 1dt + \int_{t=1}^{\infty} e^{-st}\cdot 0dt$$

$$(s^2+4)\bar{y}(s) - 5.0 - 1 = \left.\frac{e^{-st}}{-s}\right|_{t=0}^{1} + 0 = \frac{-1}{s}\left[e^{-s}-1\right] = \frac{1-e^{-s}}{s}$$

$$\therefore (s^2+4)\bar{y}(s) = 1 + \frac{1}{s} - \frac{1}{s}e^{-s}$$

$$\therefore \bar{y}(s) = \frac{1}{s^2+4} + \frac{1}{s(s^2+4)} - e^{-s}\frac{1}{s(s^2+4)}$$

$$\bar{y}(s) = \frac{1}{2}\left(\frac{2}{s^2+2^2}\right) + \frac{1}{4}\left[\frac{1}{s} - \frac{s}{s^2+2^2}\right] - e^{-s}\frac{1}{4}\left[\frac{1}{s} - \frac{s}{s^2+2^2}\right] \quad \dots (1)$$

Let $\bar{f}(s) = \frac{1}{4}\left[\frac{1}{s} - \frac{s}{s^2+2^2}\right]$

$$\therefore f(t) = L^{-1}\bar{f}(s) = \frac{1}{4}\left[L^{-1}\left(\frac{1}{s}\right) - L^{-1}\frac{s}{s^2+2^2}\right] = \frac{1}{4}\left[1-\cos 2t\right]$$

and $L^{-1}\left[e^{(-s)}\bar{f}(s)\right] = H(t-1)f(t-1)$

$\therefore$ from (1)

$$L^{-1}\left[\bar{y}(s)\right] = \frac{1}{2}L^{-1}\left(\frac{2}{s^2+2^2}\right) + \frac{1}{4}L^{-1}\left(\frac{1}{s} - \frac{s}{s^2+2^2}\right) + L^{-1}\left[e^{-s}\frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+2^2}\right)\right]$$

$$\therefore y(t) = \frac{1}{2}\sin(2t) + \frac{1}{4}(1-\cos 2t) + H(t-1)\frac{1}{4}\left[1-\cos 2(t-1)\right]$$

Q.4(a) **Find the orthogonal Trajectory of $3x^2 - 3y^2 + x^3 - 3xy^2$ = constant.** **[6]**

(A) $u = 3x^2 - 3y^2 + x^3 - 3xy^2$ = constant

$u_x = 6x + 3x^2 - 3y^2$, $u_y = -6y - 6xy$

$dv = V_x dx + V_y dy$

$= -u_y dx + u_x dy$

$-(-6y - 6xy)\,dx + (6x + 3x^2 - 3y^2)\,dy$

which is exact D.E.

$\therefore \quad \int dv = \underset{y=\text{cont}}{\int (6y+6xy)dx} + \underset{\text{free from } x}{\int (6x+3x^2-3y^2)dy} + c$

$v = 6xy + 3x^2y - y^3 + c$ is orthogonal trajectory.

Q.4(b) Using Greens theorem **[6]**

Evaluate $\oint_C \left((2xy)dx - (y^2)dy\right)$

where C is boundary of the region bounded by $3x^2 + 4y^2 = 12$.

(A) On equating $(2xy)dx - (y^2)\,dy$ with $Mdx + Ndy$

we get $M = 2xy \qquad N = -y^2$

$\therefore \ \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0 - 2x = -2x$ … (1)

equation of curve is $3x^2 + 4y^2 = 12$

$$\therefore \ \frac{x^2}{4} + \frac{y^2}{3} = 1 \quad \text{i.e.} \left(\frac{x}{2}\right)^2 + \left(\frac{y}{\sqrt{3}}\right)^2 = 1$$

Let $\frac{x}{2} = r\cos\theta,\ \frac{y}{\sqrt{3}} = r\sin\theta \ \therefore\ dxdy = 2\sqrt{3}\ r\,dr\,d\theta$

By green Theorem

$$\oint 2xy\,dx - y^2dy = \int_c Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

$$= \iint_R (-2x)\,dxdy \qquad = -2\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r\cos\theta \cdot 2\sqrt{3}\,r\,dr\,d\theta$$

$$= -4\sqrt{3}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^2\cos\theta\,dr\,d\theta \qquad = -4\sqrt{3}\left.\frac{r^3}{3}\right|_0^1 \cdot \sin\theta\Big|_{\theta=0}^{2\pi}$$

$$= \frac{-4\sqrt{3}}{3}(1-0)\times(0-0) = 0$$

Thus $\oint_c 2xy\,dx - y^2dy = 0$

Q.4(c) Find the Fourier integral representation of **[8]**

$$f(x) = \begin{cases} 1 - x^2 & \text{for } |x| \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

Hence prove that $\displaystyle\int_{\lambda=0}^{\infty} \frac{\cos\left(\frac{\lambda}{2}\right)(\sin\lambda - \lambda\cos\lambda)}{\lambda^3}\,d\lambda = \frac{3\pi}{16}$.

(A) $\therefore\ f(x) = \begin{cases} 1 - x^2 & \text{for } |x| \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$ is even for slow in$(-\infty, +\infty)$

∴ Its Fourier integral is

$$f(x) = \frac{2}{\pi}\int_{\lambda=0}^{\infty}\int_{u=0}^{\infty} f(u)\cos(\lambda u)\cos(\lambda x)\,du\,dx$$

$$f(x) = \frac{2}{\pi}\int_{\lambda=0}^{\infty}\left[\int_{u=0}^{\infty} f(u)\cos(\lambda u)\,du\right]\cos(\lambda x)\,d\lambda$$

$$= \frac{2}{\pi}\int_{\lambda=0}^{\infty}\left[\int_{u=0}^{\infty} f(u)\cos\lambda u\,du + \int_{u=1}^{\infty} f(x)\cos\lambda u\,du\right]\cos(\lambda x)\,d\lambda$$

$$= \frac{2}{\pi}\int_{\lambda=0}^{\infty}\left[\int_{\lambda=0}^{1} (1-u^2)\cos\lambda u\,du + \int_{u=1}^{\infty} o\cos\lambda u\,du\right]\cos(\lambda x)\,d\lambda$$

$$= \frac{2}{\pi}\int_{\lambda=0}^{\infty}\left[\left[(1\text{-}u^2)\left(\frac{\sin\lambda u}{\lambda}\right)\text{-}(\text{-}2u)\left(\frac{\text{-}\cos(\lambda u)}{\lambda^2}\right)\right.\right.$$

$$\left.\left. + (\text{-}2)\left(\frac{\text{-}\sin(\lambda u)}{\lambda^3}\right)\right]_{u=0}^{1} + 0\right]\cos(\lambda x)\,d\lambda$$

$$= \frac{2}{\pi}\int_{\lambda=0}^{\infty}\left[\frac{1}{\lambda}(1-u^2)\sin\lambda u - \frac{2}{\lambda^2}u\cos(\lambda u) + \frac{2}{\lambda^3}\sin(\lambda u)\right]_{u=0}^{1}\cos(\lambda x)\,d\lambda$$

$$= \frac{2}{\pi}\int_{\lambda=0}^{\infty}\left[\frac{1}{\lambda}(0-0) - \frac{2}{\lambda^2}[1\cdot\cos\lambda - 0] + \frac{2}{\lambda^3}[\sin\lambda - 0]\right]\cos(\lambda x)\,d\lambda$$

$$\therefore\quad f(x) = \frac{2}{\pi}\int_{\lambda=0}^{\infty}\left[\frac{2\sin\lambda}{\lambda^3} - \frac{2\cos\lambda}{\lambda^2}\right]\cos(\lambda x)\,d\lambda$$

$$\therefore\quad f(x) = \frac{4}{\pi}\int_{\lambda=0}^{\infty}\frac{(\sin\lambda - \lambda\cos\lambda)}{\lambda^3}\cos(\lambda x)\,d\lambda$$

be the required Fourier integral

Now put $x = \frac{1}{2}$

$$\therefore\quad \int_{\lambda=0}^{\infty}\left(\frac{\sin\lambda - \lambda\cos\lambda}{\lambda^3}\right)\cos\left(\frac{\lambda}{2}\right)d\lambda = \frac{\pi}{4}f\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{4}\cdot\left(1 - \frac{1}{4}\right)$$

$$\therefore\quad \int_{\lambda=0}^{\infty}\left(\frac{\sin\lambda - \lambda\cos\lambda}{\lambda^3}\right)\cos\left(\frac{\lambda}{2}\right)d\lambda = \frac{3\pi}{16}$$

put $\lambda = x,\ d\lambda = dx$

$$\therefore\quad \int_{x=0}^{\infty}\left(\frac{\sin x - x\cos x}{x^3}\right)\cos\left(\frac{x}{2}\right)dx = \frac{3\pi}{16}$$

Q.5(a) Find Inverse Laplace Transform of $\dfrac{s}{s^4+8s^2+16}$ using Convolution theorem. **[6]**

(A) $$\frac{s}{s^4+8s^2+16} = \frac{s}{\left(s^2+4\right)^2} = \left(\frac{s}{s^2+4}\right)\left(\frac{1}{s^2+4}\right) = \bar{f}_1(s)\cdot\bar{f}_2(s) \quad \text{... (1)}$$

where $\bar{f}_1(s) = \dfrac{s}{s^2+4} = \dfrac{s}{s^2+2^2}$ and $\bar{f}_2(s) = \dfrac{1}{s^2+4} = \dfrac{1}{2}\dfrac{2}{s^2+2^2}$

$\therefore$ $f_1(t) = \cos(2t)$ and $f_2(t) = \dfrac{1}{2}\sin(2t)$

$\therefore$ By Convolution Theorem

$$\therefore L^{-1}\left[\frac{s}{s^4+8s^2+16}\right] = L^{-1}\left[\bar{f}_1(s)\cdot\bar{f}_2(s)\right] = \int_{u=0}^{t} f_1(u)\cdot f_2(t-u)\,du$$

$$= \int_{u=0}^{t} \cos(2u)\cdot\frac{1}{2}\sin(2t-2u)\cdot du$$

$$= \frac{1}{2}\int_{u=0}^{t} \sin(2t-2u)\cos(2u)\,du$$

$$= \frac{1}{2}\int_{u=0}^{t} \frac{1}{2}\left[\sin(2t)+\sin(2t-4u)\right]du$$

$$= \frac{1}{4}\left[\sin(2t)u - \frac{\cos(2t-4u)}{-4}\right]_{u=0}^{t}$$

$$= \frac{1}{4}\left[\sin(2t)\left[t-0\right] + \frac{1}{4}\left[\cos(-2t)-\cos(2t)\right]\right]$$

$$= \frac{1}{4}\left[t\sin(2t) + \frac{1}{4}\left[\cos(2t)-\cos(2t)\right]\right]$$

$$\therefore L^{-1}\left[\frac{s}{s^4+8s^2+16}\right] = \frac{t}{4}\sin(2t)$$

Q.5(b) Find the Bilinear Transform which transform the points z = 2, i, -2 of z-plane into the points w = 1, i, -1 of the w-plane respectively. Also find fixed points of this transformation. **[6]**

(A) $z_1 = 2$, $z_2 = i$, $z_3 = -2$ and $w_1 = 1$ $w_2 = i$ and $w_3 = -1$

we know Bilinear Transform is

$$\frac{w-w_1}{w_1-w_2}\cdot\frac{w_2-w_3}{w_3-w} = \frac{z-z_1}{z_1-z_2}\cdot\frac{z_2-z_3}{z_3-z}$$

$$\therefore \quad \frac{w-1}{1-i}\cdot\frac{i+1}{-1-w} = \frac{z-2}{z-i}\cdot\frac{i+2}{-2-z}$$

$$\therefore \quad \frac{w-1}{1+w}\cdot\frac{1+i}{-1+i} = \frac{z-2}{z+2}\,\frac{2+i}{-2+i}$$

$$\therefore \quad \frac{w-1}{1+w} = \frac{z-2}{z+2}\cdot\frac{2+i}{-2+i}\times\frac{-1+i}{1+i} = \frac{(z-2)}{(z+2)}\cdot\frac{(4-3i)}{5}$$

$$\frac{w-1}{w+1} = \frac{4z-8-3iz+6i}{5z+10}$$

By compondant dividendo

$$\frac{w-1+w+1}{w-1-(w+1)} = \frac{4z-8-3iz+6i+5z+10}{4z-8-3iz+6i-5z-10}$$

$$\frac{2z}{-2} = \frac{9z+2-3iz+6i}{-z-18-3iz+6i} = \frac{(9-3i)z+(2+6i)}{-(1+zi)z-(18-6i)}$$

$$\therefore \quad w = \frac{(9+3i)z+(2+6i)}{(1+3i)z-(18-6i)} = \frac{3z(3-i)+(6i-2i^2)}{z(1+3i)+(6i-18)}$$

$$w = \frac{3z(3-i)+2i(3-i)}{z(1+3i)+(6i+18i^2)} = \frac{(3-i)(3z+2i)}{z(1+3i)+6i(1+3i)}$$

$$w = \frac{(3-i)(3z+2i)}{(1+3i)(z+6i)} = \frac{(3-i)(3z+2i)}{(-i^2+3i)(z+6i)} = \frac{(3-i)(3z+2i)}{i(z+6i)(3-i)}$$

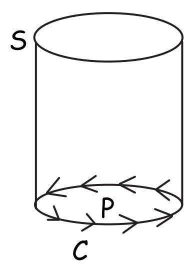
$$\therefore \quad w = \frac{3z+2i}{iz+6}$$

Q.5(c) Find $\iint_s \left[(\nabla\times\bar{F})\cdot\hat{n}\right]ds$, **[8]**

where $\bar{F}$ =(2x–y+z)i + (x+y–z^2)j + (3x–2y+4z)k
over the surface of the cylinder x^2 + y^2 = 4 bounded by z = 9
and open at the end z = 0.

(A) From figure and stokes Theorem

$$\iint_s(\nabla\times\bar{F}\cdot\hat{n})ds = \oint_C \bar{F}\,.\,d\bar{r} = \iint_P\left(\nabla\times\bar{F}\cdot\hat{n}\right).ds \qquad ...(1)$$



C: $x^2 + y^2 = 4$
ie C: $x^2 + y^2 = 2^2$

Now

$$\nabla\times\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y+z & x+y-z^2 & 3x-2y+4z \end{vmatrix}$$

$$\nabla\times\bar{F} = i(-2-(2z))+j(z-1)+k(1(-1)) = (2z-2)\ i+2j+2k$$

$$\therefore\ \nabla\times\bar{F}\cdot\hat{n} = [2(z-1)\ i + 2j + 2k]\cdot k = 2$$

$\therefore \quad \iint_P \nabla \times \bar{F} \cdot \hat{n}\, ds = \iint_P 2\, ds = 2\iint_P ds = 2 \text{ area of P}$

$$= 2 \times \pi r^2 = 2 \times \pi \cdot 4 = 8\pi \qquad \text{...(2)}$$

$\therefore$ From (1) and (2)

$$\iint_s (\nabla \times \bar{F} \cdot \hat{n}\, ds = 8\pi$$

Q.6(a) Find the directional derivative of $\phi = xy^2 + yz^2$ at (2, −1, 1) along the line 2(x − 2) = y + 1 = z − 1. [6]

(A) $\phi = xy^2 + yz^2$

$\therefore \nabla\phi = iy^2 + j(2xy + z^2) + k(2yz)$

$\nabla\phi|_{(2, -1, 1)} = i(1) + j(-4 + 1) + k(-2)$

$\nabla\phi|_{(2, -1, 1)} = 1i - 3j - 2k \qquad \text{... (1)}$

By given equation of line is 2(x-2) = (y + 1)= z-1

i.e. $\dfrac{x-2}{1} = \dfrac{y-(-1)}{2} = \dfrac{z-1}{2} \qquad \text{... (2)}$

$\therefore \quad \bar{a} = 1i + 2j + 2k$ be any vector along line $\qquad$ ... (3)

$|z| = \sqrt{1+4+4} = \sqrt{9} = 3$

$\therefore \quad \hat{a} = \dfrac{\bar{a}}{|a^{-1}|} = \dfrac{1}{3}(1i + 2j + 2k)$

$\therefore$ Directional derivative of ϕ at (2,−1, 1) along the line 2(x−2) = y + 1 = z − 1

$$= \nabla\phi \cdot \hat{a} \pm (i - 3j - 2k) \cdot \frac{1}{3}(i + 2j + 2k)$$

$$= \frac{1}{3}(1 - 6 - 4) = \frac{-9}{3} = -3$$

Q.6(b) Obtain the complex form of Fourier series for the function $f(x) = e^{4x}$ in 0 < x < 4. [6]

(A) We know complex form of Fourier series in c to c + 2ℓ is

$$f(x) = \sum_{n=-\infty}^{\infty} An\, e^{i\left(\frac{n\pi x}{\ell}\right)} \quad \text{where} \quad An = \frac{1}{2\ell}\int_{x=c}^{c+2\ell} f(x) e^{-i\left(\frac{n\pi x}{\ell}\right)} d_x$$

On equating the interval (0, 4) with (c, c +2ℓ)

c = 0, c +2ℓ = 4, u = 4 $\therefore$ ℓ = 2

$\therefore$ Complex form of Fourier series (0, 4) is

$$f(x) = \sum_{n=-\infty}^{\infty} An\, e^{i\left(\frac{n\pi x}{2}\right)} \qquad \text{... (1)} \qquad \text{where}$$

$$An = \frac{1}{2 \cdot 2}\int_{x=0}^{4} f(x) e^{-i\left(\frac{n\pi x}{2}\right)} dx = \frac{1}{4}\int_{x=0}^{4} e^{4x} e^{-i\left(\frac{n\pi x}{2}\right)} dx$$

$$\therefore An = \frac{1}{4}\int_{x=0}^{4} e^{(8-in\pi)\frac{x}{2}} dx = \frac{1}{4}\left.\frac{e^{(8-in\pi)\frac{x}{2}}}{(8-in\pi)\frac{1}{2}}\right|_{x=0}^{4}$$

$$= \frac{1}{2}\frac{8+in\pi}{(8-in\pi)(8+in\pi)}\left[e^{2(8-in\pi)} - e^{0}\right]$$

$$= \frac{1}{2}\frac{8+in\pi}{64+n^2\pi^2}\left[e^{16}e^{-2in\pi} - 1\right] = \frac{8+in\pi}{2(64+n^2\pi^2)}\left(e^{16}-1\right) \quad [\because\ e^{\pm in\pi}\ (-1)^n]$$

$$= \frac{e^4(8+in\pi)}{64+n^2\pi^2}\left(\frac{e^4-e^{-4}}{2}\right) = \frac{e^4(8+in\pi)}{64+4^2\pi^2}\sinh(4) \qquad \ldots (2)$$

From (1) and (2)

$$F(x) = \sum_{n=-\infty}^{\infty}\frac{e^4(8+in\pi)\sinh(4)}{64+n^2\pi^2}e^{i\left(\frac{n\pi x}{2}\right)}$$

$$\therefore\ f(x) = e^4\sinh(4)\sum_{n=-\infty}^{\infty}\left[\frac{8+in\pi}{64+n^2\pi^2}\right]e^{i\left[\frac{n\pi x}{2}\right]}$$

Q.6(c) Find half range sine series of the function **[8]**
$f(x) = x(3 - x)$ in $0 \le x \le 3$
Hence prove that

(i) $\frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\ldots = \frac{\pi^6}{960}$ **(ii)** $\sum_{n=1}^{\infty}\frac{1}{(n)^6} = \frac{\pi^6}{945}$

(A) We know half range sine series in (0, 3) is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{(n\pi x)}{3} \qquad \ldots(1) \quad \text{where}$$

$$b_n = \frac{2}{3}\int_{x=0}^{3} f(x)\sin\left(\frac{n\pi x}{3}\right)dx = \frac{2}{3}\int_{x=0}^{3}(3x-x^2)\sin\left(\frac{n\pi x}{3}\right)dx$$

$$b_n = \frac{2}{3}\left[(3x-x^2)\left[\frac{-\cos(n\pi x/9)}{n\pi/3}\right] - (3-2x)\left[\frac{-\sin(n\pi x/3)}{(n\pi/3)^2}\right] + (-2)\frac{\cos(n\pi x/3)}{(n\pi/3)}\right]_{x=0}^{3}$$

$$b_n = \frac{2}{3}\left[\frac{-3}{n\pi}(3x-x^2)\cos\left(\frac{n\pi x}{3}\right) + \frac{9}{n^2\pi^2}(3-2x)\sin\left(\frac{n\pi x}{3}\right) - 2\times\frac{27}{n^3\pi^3}\cos\left(\frac{n\pi x}{3}\right)\right]_{x=0}^{3}$$

$$b_n = \frac{2}{3}\left[\frac{-3}{n\pi}[0-0] + \frac{9}{n^2\pi^2}[0-0] - \frac{54}{n^3\pi^3}\left[(-1)^n - 1\right]\right]$$

$$b_n = \frac{36\left(1-(-1)^n\right)}{n^3\pi^3} \qquad \ldots(2)$$

From (1) and (2)

$$F(x) = \sum_{n=1}^{\infty}\frac{36(1-(-1)^n)}{n^3\pi^3}\cdot\sin\left(\frac{n\pi x}{3}\right) = \frac{36}{\pi^3}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^3}\right]\sin\left(\frac{n\pi x}{3}\right)$$

Now Parse Val's identity of (1) is

$$\sum_{n=1}^{\infty}(b_n)^2 = \frac{2}{3}\int_{x=0}^{3}\left[f(x)\right]^2 dx = \frac{2}{3}\int_{x=0}^{3} x^2\left(9-\sigma x+x^2\right)dx$$

$$\sum_{n=1}^{\infty}\frac{36\times 36}{n^6\pi^6}\left[1+1-2(-1)^n\right] = \frac{2}{3}\int_{x=0}^{3} 9x^2-\sigma x^3+x^4\,dx$$

$$\frac{36\times 36}{\pi^6}\sum_{n\pi}^{\infty}\frac{1-(-1)^n}{n^6} = \frac{1}{3}\left[\frac{9x^3}{3}-\frac{\sigma x^4}{4}+\frac{x^5}{5}\right]_{x=0}^{3}$$

$$\frac{36\times 36}{\pi^6}\left[\frac{2}{1^6}+a\ell\frac{2}{3^6}+a\ell\frac{2}{5^6}\ldots\right] = \frac{1}{3}\left[\frac{95}{3}-\frac{1}{2}3^4+\frac{1}{5}\cdot 35\right]$$

$$\frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}\ldots = \frac{\pi^6}{36\times 36\times 2}\times\frac{9\times 9}{3}\left[1-\frac{3}{2}+\frac{3}{5}\right]$$

$$= \frac{\pi^6}{2\times 48}\times\frac{10-15+16}{10} = \frac{\pi^6\cdot 1}{960}$$

$$\therefore\quad \frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}\ldots = \frac{\pi^6}{960} \qquad \ldots(3)$$

$$\text{Let } s = \sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\frac{1}{6^6}+\ldots \qquad \ldots(4)$$

$$s = \left(\frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}+\ldots\right)+\left(\frac{1}{(2\times 1)^6}+\frac{1}{(2\times 2)^6}+\frac{1}{(2\times 3)^6}+\ldots\right)$$

$$s = \frac{\pi^6}{960}+\frac{1}{64}\left[\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}\ldots\right]$$

$$s = \frac{\pi^6}{960}+\frac{1}{64}\times 5$$

$$s-\frac{s}{64} = \frac{\pi^6}{960}$$

$$\frac{63}{64}s = \frac{\pi^6}{960}$$

$$\therefore\quad s = \frac{\pi^6}{960}\times\frac{64}{63} = \frac{\pi^6}{15\times 63} = \frac{\pi^6}{945}$$

$$\therefore\quad \sum_{n\pi}^{\infty}\left(\frac{1}{n^6}\right) = \frac{\pi^6}{945}$$

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