

S.E. Sem. III [BIOM]
Applied Mathematics III
 Prelim Question Paper Solution



Time : 3 Hrs.]

[Marks : 80

Q.1(a) Find Laplace Transformation of $\frac{\sin(3t)}{t}$ [5]

(A) We know $L\{\sin 3t\} = \frac{3}{s^2 + 3^2}$

$$\therefore L\left[\frac{\sin(3t)}{t}\right] = \int_{s=3}^{\infty} \frac{3}{s^2 + 3^2} ds = \frac{3}{3} \tan^{-1}(s/3) \Big|_{s=3}^{\infty}$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s/3) = \frac{\pi}{2} - \tan^{-1}(s/3)$$

$$\therefore L\left[\frac{\sin(3t)}{t}\right] = \cot^{-1}(s/3)$$

Q.1(b) Prove that $f(z) = \cosh z$ is analytic and find its derivative. [5]

(A) $f(z) = \cosh(z) = \cosh(x+iy) = \cosh x \cos y + i \sinh x \sin y = u + iv$

$$\therefore u = \cosh x \cos y, \quad v = \sinh x \sin y$$

$$u_x = \sinh x \cos y, \quad u_y = -\cosh x \sin y, \quad v_x = \cosh x \sin y, \quad v_y = \sinh x \cos y$$

$$\therefore u_x = v_y \quad \text{and} \quad u_y = -v_x$$

$\therefore u$ and v satisfies C.R. equations

$\therefore f(z) = u + iv$ is analytic

$$\text{Now } ux|_{(z,0)} = \sinh z \quad vx|_{(z,0)} = 0$$

$$f'(z) = u_x + iv_x$$

$$= \sinh x \cos y + i \cosh x \sin y$$

put $x = z, \quad y = 0$

$$f'(z) = \sinh z$$

$$\therefore f'(z)|_{(z,0)} = ux|_{(z,0)} + ivx|_{(z,0)}$$

$$= \sinh z + i0 = \sinh z$$

Q.1(c) Find Fourier series for $f(x) = 16 - x^2$ over $(-4, +4)$ [5]

(A) $\therefore f(x)$ is even function in $(-4, 4)$ \therefore its Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{4}\right) \quad \dots (1)$$

$$\text{where } a_0 = \frac{2}{4} \int_{x=0}^4 f(x) dx = \frac{1}{2} \int_{x=0}^4 (16 - x^2) dx = \frac{1}{2} \left[16x - \frac{x^3}{3} \right]_0^4 = \frac{1}{2} \left[64 - \frac{64}{3} \right]$$

$$\frac{32}{3} \times 2 = \frac{64}{3} \quad \dots (2)$$

$$a_n = \frac{2}{4} \int_{x=0}^4 f(x) \cos\left(\frac{n\pi x}{4}\right) dx = \frac{1}{2} \int_{x=0}^4 (16 - x^2) \cos\left(\frac{n\pi x}{4}\right) dx$$

$$a_n = \frac{1}{2} \left[(16 - x^2) \frac{\sin(n\pi x / 4)}{n\pi / 4} - (-2x) \left(\frac{-\cos(n\pi x / 4)}{(n\pi / 4)^2} \right) + (-2) \left(\frac{-\sin(n\pi x / 4)}{(n\pi / 4)^3} \right) \right]_{x=0}^4$$

$$a_n = \frac{1}{2} \left[\frac{4}{n\pi} (16 - x^2) \sin\left(\frac{n\pi x}{4}\right) - \frac{16}{n^2 \pi^2} \times 2 \times x \cos\left(\frac{n\pi x}{4}\right) + \frac{64}{n^3 \pi^3} \times 2 \sin\left(\frac{n\pi x}{4}\right) \right]_{x=0}^4$$

$$a_n = \frac{1}{2} \left[\frac{4}{n\pi} (0 - 0) - \frac{32}{n^2 \pi^2} [4(-1)^n - 0] + \frac{128}{n^3 \pi^3} [0 - 0] \right] = \frac{-1}{2} \frac{32}{n^2 \pi^2} 4(-1)^n$$

$$\therefore a_n = \frac{-64}{n^2 \pi^2} (-1)^n \quad \dots (3)$$

From (1), (2), (3)

$$f(x) = \frac{1}{2} \left(\frac{64}{3} \right) + \sum_{n=1}^{\infty} \frac{-64}{n^2 \pi^2} (-1)^n \cos\left(\frac{n\pi x}{4}\right)$$

$$\therefore f(x) = \frac{32}{2} - \frac{64}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{n\pi x}{4}\right)$$

Q.1(d) Prove that $J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin(x)}{x} - \cos(x) \right)$ [5]

(A) We know $2n J_n(x) = x [J_{n-1}(x) + J_{n+1}(x)]$

$$\therefore J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

Put $n = \frac{1}{2}$

$$\therefore n + 1 = \frac{3}{2} \quad \text{and} \quad n - 1 = -\frac{1}{2}$$

$$J_{\frac{3}{2}}(x) = \frac{2 - \left(\frac{1}{2}\right)}{x} J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x)$$

$$= \frac{1}{x} \sqrt{\frac{2}{\pi x}} \sin x - \sqrt{\frac{2}{\pi x}} \cos x$$

$$\therefore J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left[\frac{\sin x}{x} - \cos x \right]$$

Q.2(a) Show that the vector field $\vec{F} = (3x^2 y)\mathbf{i} + (x^3 - 2yz^2)\mathbf{j} + (3z^2 - 2y^2z)\mathbf{k}$ is [6] conservative and find $\phi(x, y, z)$ such that $\vec{F} = \Delta\phi$. Also evaluate the line integral $\int \vec{F} \cdot d\vec{r}$ from $(2, 1, 1)$ to $(2, 0, 1)$

$$(A) \quad \nabla \times \vec{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y & x^3 - 2yz^2 & 3z^2 - 2y^2z \end{vmatrix} = \begin{matrix} \mathbf{i}(-4yz - (-4yz)) \\ -\mathbf{j}(0 - 0) \\ +\mathbf{k}(3x^2 - 3x^2) \end{matrix}$$

$$\therefore \nabla \times \vec{F} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \vec{0}$$

$\therefore \vec{F}$ is irrotational

\therefore There exist scalar point function ϕ such that

$$\vec{F} = \nabla\phi$$

$$\therefore \phi = \phi(x, y, z)$$

$$\therefore d\phi = \nabla\phi \cdot d\vec{z} = \vec{F} \cdot d\vec{z} = (3x^2y)dx + (x^3 - 2yz^2)dy + (3z^2 - 2y^2z)dz$$

$$\therefore \phi(x, y, z) = \int (3x^2y)dx + (x^3 - 2yz^2)dy + (3z^2 - 2y^2z)dz$$

$$\therefore \phi(x, y, z) = \frac{3x^3}{3}y + \left(\frac{-2y^2}{2}z^2\right) + z^3 + c$$

$\therefore \phi(x, y, z) = x^3y - y^2z^2 + z^3 + c$ be the required scalar point function.

$$\begin{aligned} \therefore \int_{(2,1,1)}^{(2,0,1)} \vec{F} \cdot d\vec{z} &= \int_{(2,1,1)}^{(2,0,1)} [(3x^2y)dx + (x^3 - 2yz^2)dy + (3z^2 - 2y^2z)dz] \\ &= x^3y - y^2z^2 + z^3 + c \Big|_{(2,1,1)}^{(2,0,1)} = 1 - [8 - 1 + 1] = -7 \end{aligned}$$

$$\therefore \int_{(2,1,1)}^{(2,0,1)} \vec{F} \cdot d\vec{z} = -7$$

Q.2(b) Find the Fourier series for $f(x) = \frac{x - \pi}{4}$; $0 \leq x \leq 2\pi$. [6]

Hence prove that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$

(A) We know fourier series for the function in $[0, 2\pi]$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) \quad \dots (1) \text{ where}$$

$$a_0 = \frac{1}{\pi} \int_{x=0}^{2\pi} f(x) dx, \quad a_n = \frac{1}{\pi} \int_{x=0}^{2\pi} f(x) \cos(nx) dx, \quad b_n = \frac{1}{\pi} \int_{x=0}^{2\pi} f(x) \sin(nx) dx$$

$$a_0 = \frac{1}{\pi} \int_{x=0}^{2\pi} \frac{1}{4}(x - \pi) dx = \frac{1}{4\pi} \left. \frac{(x - \pi)^2}{2} \right|_0^{2\pi} = \frac{1}{8\pi} [\pi^2 - (-\pi)^2] = 0 \quad \dots (2)$$

$$a_n = \frac{1}{\pi} \int_{x=0}^{2\pi} \left(\frac{x-\pi}{4} \right) \cos(nx) dx = \frac{1}{4\pi} \left[(x-\pi) \left[\frac{\sin(nx)}{n} \right] - 1 \left[\frac{-\cos(nx)}{n^2} \right] \right]_{x=0}^{2\pi}$$

$$\therefore a_n = \frac{1}{4\pi} \left[\frac{1}{n} (x-\pi) \sin(nx) + \frac{1}{n^2} [\cos(nx)] \right]_{x=0}^{2\pi}$$

$$a_n = \frac{1}{4\pi} \left[\frac{1}{n} [0-0] + \frac{1}{n^2} [1-1] \right] = 0 \quad \dots (3)$$

$$b_n = \frac{1}{\pi} \int_{x=0}^{2\pi} \left(\frac{x-\pi}{4} \right) \sin(nx) dx = \frac{1}{4\pi} \left[(x-\pi) \left(\frac{-\cos(nx)}{n} \right) - 1 \left(\frac{-\sin(nx)}{n^2} \right) \right]_{x=0}^{2\pi}$$

$$b_n = \frac{1}{4\pi} \left[-\frac{1}{n} (x-\pi) \cos(nx) + \frac{1}{n^2} \sin(nx) \right]_{x=0}^{2\pi}$$

$$= \frac{1}{4\pi} \left[-\frac{1}{n} [\pi \cdot 1 - (-\pi) \cdot 1] + \frac{1}{n^2} (0-0) \right]$$

$$b_n = \frac{1}{4\pi} \left[-\frac{1}{n} (2\pi) \right] = \frac{-1}{4\pi} \cdot \frac{2\pi}{n} = \frac{-1}{2n} \quad \dots(4)$$

From (1), (2), (3) and (4)

$$f(x) = \frac{1}{2} \cdot 0 + \sum_{n=1}^{\infty} 0 \cdot \cos(nx) + \sum_{n=1}^{\infty} \left(\frac{-1}{2n} \right) \sin(nx)$$

$$\therefore f(x) = -\frac{1}{2} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$$

$$\therefore \frac{1}{4} (x-\pi) = -\frac{1}{2} \sum_{n=1}^{\infty} \left[\frac{\sin(nx)}{n} \right], \quad 0 \leq x \leq 2\pi$$

Put $x = \pi/2$

$$\frac{1}{4} \left(\frac{\pi}{2} - \pi \right) = \frac{-1}{2} \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n}$$

$$\therefore \frac{1}{4} \left(\frac{-\pi}{2} \right) = \frac{1}{2} \left[1 + 0 - \frac{1}{3} + 0 + \frac{1}{5} + 0 - \frac{1}{7} \dots \right]$$

$$\therefore \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Q.2(c) Find Inverse Laplace transform of

[8]

$$(i) \frac{s+19}{(s+9)(s^2+4)}, \quad (ii) \frac{e^{-3s}}{(s^2+10s+29)}$$

(A) (i) $\frac{s+19}{(s+9)(s^2+4)}$

$$\text{Let } \frac{s+19}{(s+9)(s^2+4)} = \frac{A}{s+9} + \frac{Bs+C}{s^2+4} \quad \dots (1)$$

$$\therefore A(s^2+4) + (Bs+C)(s+9) = s+19 \quad \dots (2)$$

$$\text{if } s = -9, \quad 85A = +10 \quad \therefore A = \frac{2}{17}$$

$$\text{if } s = 0 \quad 4A + 9C = 19$$

$$9C = 19 - 4A = 19 - \frac{8}{17} = \frac{315}{17}$$

$$\therefore C = \frac{35}{17}$$

$$\text{if } s = 1, \quad 5A + 10(B+C) = 20$$

$$B+C = \frac{1}{10} [20 - 5A] = \frac{1}{10} \left[20 - \frac{10}{17} \right] = \left[2 - \frac{1}{17} \right] = \frac{33}{17}$$

$$B = \frac{33}{17} - C = \frac{33}{17} - \frac{35}{17}$$

$$B = \frac{-2}{17}$$

$$\therefore \frac{s+19}{(s+9)(s^2+4)} = \frac{2}{17} \cdot \frac{1}{s+9} + \frac{-2}{17} \cdot \frac{s}{s^2+4} + \frac{35}{17} \cdot \frac{1}{s^2+4}$$

$$= \frac{1}{17} \left[2 \cdot \frac{1}{s+9} + \frac{35}{2} \left(\frac{2}{s^2+2^2} \right) - \frac{2s}{s^2+2^2} \right]$$

$$\therefore L^{-1} \left[\frac{s+19}{(s+9)(s^2+4)} \right] = \frac{1}{17} \left[2L^{-1} \left(\frac{1}{s+9} \right) + \frac{35}{2} L^{-1} \left(\frac{2}{s^2+2^2} \right) - 2L^{-1} \left(\frac{s}{s^2+2^2} \right) \right]$$

$$= \frac{1}{17} \left[2e^{-9t} + \frac{35}{2} \sin(2t) - 2\cos(2t) \right]$$

(ii) Let $\bar{f}(s) = \frac{1}{s^2+10s+29} = \frac{1}{s^2+10s+25+4} = \frac{1}{(s+5)^2+2^2}$

$$\therefore L^{-1}[\bar{f}(s)] = L^{-1} \left[\frac{1}{(s+5)^2+2^2} \right] = e^{-5t} L^{-1} \left(\frac{1}{s^2+2^2} \right) = \frac{e^{-5t}}{2} \sin(2t)$$

$$\begin{aligned} \therefore L^{-1}\left[e^{-3s} \frac{1}{s^2 + 10s + 29}\right] &= L^{-1}\left[e^{-3s} \bar{f}(s)\right] = h(t-a)f(t-a) \\ &= h(t-3) \cdot \frac{e^{-5(t-3)}}{2} \sin[2(t-3)] \end{aligned}$$

Q.3(a) Find the analytic function $f(z) = u + iv$

[6]

$$\text{if } u + v = \frac{2 \sin(2x)}{e^{2y} + e^{-2y} - 2 \cos(2x)}$$

$$(A) \quad u + (v) = \frac{2 \sin(2x)}{2 \cosh(2y) - 2 \cos(2x)} = \frac{\sin(2x)}{\cosh(2y) - \cos(2x)} = (v)$$

$$\text{By given } f(z) = u + iv \quad \therefore i f(z) = iu - v \quad \dots (1)$$

$$\therefore (1 + i) f(z) = (u - v) + i(u + v)$$

$$\therefore F(z) = U + iV \quad \dots (2)$$

$$\text{where } F(z) = (1 + i) f(z)$$

$$u - v = U \quad \text{and } u + v = V \quad \dots (3)$$

is also analytic

$$\text{Now } F'(z) = U_{x+i} V_x = V_y + iV_x$$

$$\therefore F'(z)|_{(z,0)} = V_y|_{(z,0)} + iV_x|_{(z,0)} \quad \dots (4)$$

$$V_x = \frac{[\cosh(2y) - \cos(2x)] 2 \cos(2x) - \sin(2x) [2 \sin(2x)]}{(\cosh 2y - \cos 2x)^2}$$

$$V_x|_{(z,0)} = \frac{2[\cos 2z(1 - \cos 2z) - \sin^2(2z)]}{(1 - \cos 2z)^2} = \frac{2[\cos(2z) - \cos^2(2z) - \sin^2(2z)]}{(1 - \cos 2z)^2}$$

$$= \frac{-2[1 - \cos(2z)]}{(1 - \cos 2z)^2} = \frac{-2}{1 - \cos 2z} = \frac{-2}{2 \sin^2 z} = -\operatorname{cosec}^2 z \quad \dots (5)$$

$$V_y = \frac{-\sin(2xy)}{(\cosh 2y - \cos 2xy)} \times 2 \sinh(2y)$$

$$\therefore V_y|_{(z,0)} = \frac{-2 \sin(2y) 0}{(1 - \cos 2y)^2} = 0 \quad \dots (6)$$

Now from (4), (5) and (6)

$$F'(z)|_{(z,0)} = 0 + i(-\operatorname{cosec}^2 z) = -i \operatorname{cosec}^2(z)$$

$$\therefore F(z) = \int F'(z)|_{(z,0)} dz + c = -i \int \operatorname{cosec}^2 z dz + c = i(+\cot z) + c$$

$$\therefore (1 + i) f(z) = +i \cot z + c$$

$$\therefore f(z) = \frac{1}{1+i} i \cot z + \frac{c}{1+i} = \frac{(1-i)i}{2} \cot z + c' = \frac{(1+i)}{2} \cot z + c'$$

**Q.3(b) Show that Bessel's Fourier series in $J_2(\lambda_n x)$ for $f(x) = x^2$, [6]
($0 < x < a$) Where λ_n positive root of $J_2(ax) = 0$ is**

$$x^2 = 2a \sum_{n=1}^{\infty} \frac{J_2(\lambda_n x)}{\lambda_n J_3(\lambda_n a)}$$

(A) We know Bessel's Fourier Series is

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\lambda_i x)$$

put $n = 2$

$$\therefore f(x) = \sum_{i=1}^{\infty} c_i J_2(\lambda_i x) \quad \dots (1)$$

$$x J_2(\lambda_j x) f(x) = \sum_{i=1}^{\infty} c_i J_2(\lambda_i x) \cdot x J_2(\lambda_i x)$$

$$\int_{x=0}^a x J_2(\lambda_j x) x^2 dx = \sum_{i=1}^{\infty} c_i \int_{x=0}^a x J_2(\lambda_j x) J_2(\lambda_i x) dx$$

$$\int_{x=0}^a x^3 J_{3-1}(\lambda_j x) dx = c_j \frac{a^2}{2} [J_{2+1}(\lambda_j a)]^2$$

$$\text{Let } \lambda_j x = t \quad x = \frac{t}{\lambda_j} \quad \therefore dx = \frac{dt}{\lambda_j}$$

$$\int_{t=0}^{\lambda_j a} \frac{t^3}{(\lambda_j)^3} J_{3-1}(t) \frac{dt}{\lambda_j} = c_j \frac{a^2}{2} [J_3(\lambda_j a)]^2$$

$$\frac{1}{(\lambda_j)^4} t^3 J_3(t) \Big|_{t=0}^{\lambda_j a} = c_j \frac{a^2}{2} [J_3(\lambda_j a)]^2$$

$$\therefore c_j = \frac{2}{a^2 [J_3(\lambda_j a)]^2} \cdot \frac{1}{(\lambda_j)^4} \cdot (\lambda_j)^3 \cdot a^3 J_3(\lambda_j a)$$

$$c_j = \frac{2a}{\lambda_j J_3(\lambda_j a)} \quad \dots (2)$$

$$\therefore \text{From (1) and (2), } f(x) = \sum_{i=1}^{\infty} \frac{2a}{\lambda_i J_3(\lambda_i a)} \cdot J_2(\lambda_i x)$$

$$\therefore x^2 = 2a \sum_{i=1}^{\infty} \frac{J_2(\lambda_i x)}{\lambda_i J_3(\lambda_i a)}$$

Q.3(c) Solve the differential equation $\frac{d^2y}{dt^2} + 4y = f(t)$, with $y(0) = 0$ & [8]

$$y'(0) = 1 \text{ and } f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & 1 < t \end{cases}$$

(A) By given $y''(t) + 4y(t) = f(t)$

$$\therefore L[y''(t)] + 4Ly(t) = Lf(t) = \int_{t=0}^{\infty} e^{-st}f(t)dt = \int_{t=0}^1 e^{-st}f(t)dt + \int_{t=1}^{\infty} e^{-st}f(t)dt$$

$$s^2\bar{y}(s) - sy(0) - y'(0) + 4\bar{y}(s) = \int_{t=0}^1 e^{-st} \cdot 1dt + \int_{t=1}^{\infty} e^{-st} \cdot 0dt$$

$$(s^2 + 4)\bar{y}(s) - 5 \cdot 0 - 1 = \frac{e^{-st}}{-s} \Big|_{t=0}^1 + 0 = \frac{-1}{s} [e^{-s} - 1] = \frac{1 - e^{-s}}{s}$$

$$\therefore (s^2 + 4)\bar{y}(s) = 1 + \frac{1}{s} - \frac{1}{s}e^{-s}$$

$$\therefore \bar{y}(s) = \frac{1}{s^2 + 4} + \frac{1}{s(s^2 + 4)} - e^{-s} \frac{1}{s(s^2 + 4)}$$

$$\bar{y}(s) = \frac{1}{2} \left(\frac{2}{s^2 + 2^2} \right) + \frac{1}{4} \left[\frac{1}{s} - \frac{s}{s^2 + 2^2} \right] - e^{-s} \frac{1}{4} \left[\frac{1}{s} - \frac{s}{s^2 + 2^2} \right] \dots (1)$$

$$\text{Let } \bar{f}(s) = \frac{1}{4} \left[\frac{1}{s} - \frac{s}{s^2 + 2^2} \right]$$

$$\therefore f(t) = L^{-1}\bar{f}(s) = \frac{1}{4} \left[L^{-1} \left(\frac{1}{s} \right) - L^{-1} \left(\frac{s}{s^2 + 2^2} \right) \right] = \frac{1}{4} [1 - \cos 2t]$$

$$\text{and } L^{-1} \left[e^{(-s)} \bar{f}(s) \right] = H(t-1)f(t-1)$$

\therefore from (1)

$$L^{-1}[\bar{y}(s)] = \frac{1}{2} L^{-1} \left(\frac{2}{s^2 + 2^2} \right) + \frac{1}{4} L^{-1} \left(\frac{1}{s} - \frac{s}{s^2 + 2^2} \right) + L^{-1} \left[e^{-s} \frac{1}{4} \left(\frac{1}{s} - \frac{s}{s^2 + 2^2} \right) \right]$$

$$\therefore y(t) = \frac{1}{2} \sin(2t) + \frac{1}{4} (1 - \cos 2t) + H(t-1) \frac{1}{4} [1 - \cos 2(t-1)]$$

Q.4(a) Find the orthogonal Trajectory of $3x^2 - 3y^2 + x^3 - 3xy^2 = \text{constant}$. [6]

(A) $u = 3x^2 - 3y^2 + x^3 - 3xy^2 = \text{constant}$

$$u_x = 6x - 3x^2 - 3y^2, \quad u_y = -6y - 6xy$$

$$dv = V_x dx + V_y dy$$

$$= -u_y dx + u_x dy$$

$$= (-6y - 6xy) dx + (6x + 3x^2 - 3y^2) dy$$

which is exact D.E.

$$\therefore \int dv = \int (6y + 6xy) dx + \int (6x + 3x^2 - 3y^2) dy + c$$

$$= \int_{y=\text{cont}} (6y + 6xy)dx + \int_{\text{free from } x} (6x + 3x^2 - 3y^2)dy + c$$

$v = 6xy + 3x^2y - y^3 + c$ is orthogonal trajectory.

Q.4(b) Using Greens theorem

[6]

Evaluate $\oint_C ((2xy)dx - (y^2)dy)$

where C is boundary of the region bounded by $3x^2 + 4y^2 = 12$.

(A) On equating $(2xy)dx - (y^2)dy$ with $Mdx + Ndy$

we get $M = 2xy$ $N = -y^2$

$$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0 - 2x = -2x \quad \dots (1)$$

equation of curve is $3x^2 + 4y^2 = 12$

$$\therefore \frac{x^2}{4} + \frac{y^2}{3} = 1 \quad \text{i.e.} \left(\frac{x}{2}\right)^2 + \left(\frac{y}{\sqrt{3}}\right)^2 = 1$$

Let $\frac{x}{2} = r\cos\theta, \frac{y}{\sqrt{3}} = r\sin\theta \therefore dx dy = 2\sqrt{3} r dr d\theta$

By green Theorem

$$\begin{aligned} \oint_C 2xy dx - y^2 dy &= \int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy \\ &= \iint_R (-2x) dx dy = -2 \int_{\theta=0}^{2\pi} \int_{r=0}^1 r \cos\theta \cdot 2\sqrt{3} r dr d\theta \\ &= -4\sqrt{3} \int_{\theta=0}^{2\pi} \int_{r=0}^1 r^2 \cos\theta dr d\theta = -4\sqrt{3} \frac{r^3}{3} \Big|_0^1 \cdot \sin\theta \Big|_{\theta=0}^{2\pi} \\ &= \frac{-4\sqrt{3}}{3} (1-0) \times (0-0) = 0 \end{aligned}$$

Thus $\oint_C 2xy dx - y^2 dy = 0$

Q.4(c) Find the Fourier integral representation of

[8]

$$f(x) = \begin{cases} 1 - x^2 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

Hence prove that $\int_{\lambda=0}^{\infty} \frac{\cos\left(\frac{\lambda}{2}\right) (\sin \lambda - \lambda \cos \lambda)}{\lambda^3} d\lambda = \frac{3\pi}{16}$.

(A) $\therefore f(x) = \begin{cases} 1 - x^2 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| > 1 \end{cases}$ is even for slow in $(-\infty, +\infty)$

∴ Its Fourier integral is

$$f(x) = \frac{2}{\pi} \int_{\lambda=0}^{\infty} \int_{u=0}^{\infty} f(u) \cos(\lambda u) \cos(\lambda x) du d\lambda$$

$$f(x) = \frac{2}{\pi} \int_{\lambda=0}^{\infty} \left[\int_{u=0}^{\infty} f(u) \cos(\lambda u) du \right] \cos(\lambda x) d\lambda$$

$$= \frac{2}{\pi} \int_{\lambda=0}^{\infty} \left[\int_{u=0}^1 f(u) \cos \lambda u du + \int_{u=1}^{\infty} f(x) \cos \lambda u du \right] \cos(\lambda x) d\lambda$$

$$= \frac{2}{\pi} \int_{\lambda=0}^{\infty} \left[\int_{\lambda=0}^1 (1-u^2) \cos \lambda u du + \int_{u=1}^{\infty} 0 \cos \lambda u du \right] \cos(\lambda x) d\lambda$$

$$= \frac{2}{\pi} \int_{\lambda=0}^{\infty} \left[(1-u^2) \left(\frac{\sin \lambda u}{\lambda} \right) - (-2u) \left(\frac{-\cos(\lambda u)}{\lambda^2} \right) \right. \\ \left. + (-2) \left(\frac{-\sin(\lambda u)}{\lambda^3} \right) \right]_{u=0}^1 + 0 \cos(\lambda x) d\lambda$$

$$= \frac{2}{\pi} \int_{\lambda=0}^{\infty} \left[\frac{1}{\lambda} (1-u^2) \sin \lambda u - \frac{2}{\lambda^2} u \cos(\lambda u) + \frac{2}{\lambda^3} \sin(\lambda u) \right]_{u=0}^1 \cos(\lambda x) d\lambda$$

$$= \frac{2}{\pi} \int_{\lambda=0}^{\infty} \left[\frac{1}{\lambda} (0-0) - \frac{2}{\lambda^2} [1 \cdot \cos \lambda - 0] + \frac{2}{\lambda^3} [\sin \lambda - 0] \right] \cos(\lambda x) d\lambda$$

$$\therefore f(x) = \frac{2}{\pi} \int_{\lambda=0}^{\infty} \left[\frac{2 \sin \lambda}{\lambda^3} - \frac{2 \cos \lambda}{\lambda^2} \right] \cos(\lambda x) d\lambda$$

$$\therefore f(x) = \frac{4}{\pi} \int_{\lambda=0}^{\infty} \frac{(\sin \lambda - \lambda \cos \lambda)}{\lambda^3} \cos(\lambda x) d\lambda$$

be the required Fourier integral

Now put $x = \frac{1}{2}$

$$\therefore \int_{\lambda=0}^{\infty} \left(\frac{\sin \lambda - \lambda \cos \lambda}{\lambda^3} \right) \cos \left(\frac{\lambda}{2} \right) d\lambda = \frac{\pi}{4} f \left(\frac{1}{2} \right) \\ = \frac{\pi}{4} \cdot \left(1 - \frac{1}{4} \right)$$

$$\therefore \int_{\lambda=0}^{\infty} \left(\frac{\sin \lambda - \lambda \cos \lambda}{\lambda^3} \right) \cos \left(\frac{\lambda}{2} \right) d\lambda = \frac{3\pi}{16}$$

put $\lambda = x$, $d\lambda = dx$

$$\therefore \int_{x=0}^{\infty} \left(\frac{\sin x - x \cos x}{x^3} \right) \cos \left(\frac{x}{2} \right) dx = \frac{3\pi}{16}$$

Q.5(a) Find Inverse Laplace Transform of $\frac{s}{s^4 + 8s^2 + 16}$ using Convolution [6] theorem.

$$(A) \quad \frac{s}{s^4 + 8s^2 + 16} = \frac{s}{(s^2 + 4)^2} = \left(\frac{s}{s^2 + 4} \right) \left(\frac{1}{s^2 + 4} \right) = \bar{f}_1(s) \cdot \bar{f}_2(s) \quad \dots (1)$$

$$\text{where } \bar{f}_1(s) = \frac{s}{s^2 + 4} = \frac{s}{s^2 + 2^2} \text{ and } \bar{f}_2(s) = \frac{1}{s^2 + 4} = \frac{1}{2} \frac{2}{s^2 + 2^2}$$

$$\therefore f_1(t) = \cos(2t) \text{ and } f_2(t) = \frac{1}{2} \sin(2t)$$

\therefore By Convolution Theorem

$$\begin{aligned} \therefore L^{-1} \left[\frac{s}{s^4 + 8s^2 + 16} \right] &= L^{-1} \left[\bar{f}_1(s) \cdot \bar{f}_2(s) \right] = \int_{u=0}^t f_1(u) \cdot f_2(t-u) du \\ &= \int_{u=0}^t \cos(2u) \cdot \frac{1}{2} \sin(2t-2u) \cdot du \\ &= \frac{1}{2} \int_{u=0}^t \sin(2t-2u) \cos(2u) du \\ &= \frac{1}{2} \int_{u=0}^t \frac{1}{2} [\sin(2t) + \sin(2t-4u)] du \\ &= \frac{1}{4} \left[\sin(2t)u - \frac{\cos(2t-4u)}{-4} \right]_{u=0}^t \\ &= \frac{1}{4} \left[\sin(2t)[t-0] + \frac{1}{4} [\cos(-2t) - \cos(2t)] \right] \\ &= \frac{1}{4} \left[t \sin(2t) + \frac{1}{4} [\cos(2t) - \cos(2t)] \right] \end{aligned}$$

$$\therefore L^{-1} \left[\frac{s}{s^4 + 8s^2 + 16} \right] = \frac{t}{4} \sin(2t)$$

Q.5(b) Find the Bilinear Transform which transform the points $z = 2, i, -2$ of z -plane into the points $w = 1, i, -1$ of the w -plane respectively. Also find fixed points of this transformation. [6]

(A) $z_1 = 2, z_2 = i, z_3 = -2$ and $w_1 = 1, w_2 = i$ and $w_3 = -1$

we know Bilinear Transform is

$$\frac{w - w_1}{w_1 - w_2} \cdot \frac{w_2 - w_3}{w_3 - w} = \frac{z - z_1}{z_1 - z_2} \cdot \frac{z_2 - z_3}{z_3 - z}$$

$$\therefore \frac{w - 1}{1 - i} \cdot \frac{i + 1}{-1 - w} = \frac{z - 2}{z - i} \cdot \frac{i + 2}{-2 - z}$$

$$\begin{aligned} \therefore \frac{w-1}{1+w} \cdot \frac{1+i}{-1+i} &= \frac{z-2}{z+2} \cdot \frac{2+i}{-2+i} \\ \therefore \frac{w-1}{1+w} &= \frac{z-2}{z+2} \cdot \frac{2+i}{-2+i} \times \frac{-1+i}{1+i} = \frac{(z-2)(4-3i)}{(z+2)5} \\ \frac{w-1}{w+1} &= \frac{4z-8-3iz+6i}{5z+10} \end{aligned}$$

By compondant dividendo

$$\frac{w-1+w+1}{w-1-(w+1)} = \frac{4z-8-3iz+6i+5z+10}{4z-8-3iz+6i-5z-10}$$

$$\frac{2z}{-2} = \frac{9z+2-3iz+6i}{-z-18-3iz+6i} = \frac{(9-3i)z+(2+6i)}{-(1+zi)z-(18-6i)}$$

$$\therefore w = \frac{(9+3i)z+(2+6i)}{(1+3i)z-(18-6i)} = \frac{3z(3-i)+(6i-2i^2)}{z(1+3i)+(6i-18)}$$

$$w = \frac{3z(3-i)+2i(3-i)}{z(1+3i)+(6i+18i^2)} = \frac{(3-i)(3z+2i)}{z(1+3i)+6i(1+3i)}$$

$$w = \frac{(3-i)(3z+2i)}{(1+3i)(z+6i)} = \frac{(3-i)(3z+2i)}{(-i^2+3i)(z+6i)} = \frac{(3-i)(3z+2i)}{i(z+6i)(3-i)}$$

$$\therefore w = \frac{3z+2i}{iz+6}$$

Q.5(c) Find $\iint_S [(\nabla \times \vec{F}) \cdot \hat{n}] ds$,

[8]

where $\vec{F} = (2x-y+z)\mathbf{i} + (x+y-z^2)\mathbf{j} + (3x-2y+4z)\mathbf{k}$ over the surface of the cylinder $x^2 + y^2 = 4$ bounded by $z = 9$ and open at the end $z = 0$.

(A) From figure and Stokes Theorem

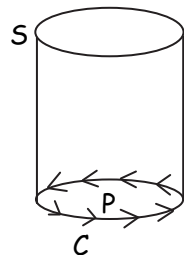
$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n} ds = \oint_C \vec{F} \cdot d\vec{r} = \iint_P (\nabla \times \vec{F}) \cdot \hat{n} ds \quad \dots(1)$$

Now

$$\nabla \times \vec{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y+z & x+y-z^2 & 3x-2y+4z \end{vmatrix}$$

$$\nabla \times \vec{F} = \mathbf{i}(-2-(2z)) + \mathbf{j}(z-1) + \mathbf{k}(1(-1)) = (2z-2)\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\therefore \nabla \times \vec{F} \cdot \hat{n} = [2(z-1)\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}] \cdot \mathbf{k} = 2$$



$C: x^2 + y^2 = 4$
ie $C: x^2 + y^2 = 2^2$

$$\begin{aligned} \therefore \iint_P \nabla \times \bar{F} \cdot \hat{n} \, ds &= \iint_P 2 \, ds = 2 \iint_P ds = 2 \text{ area of } P \\ &= 2 \times \pi r^2 = 2 \times \pi \cdot 4 = 8\pi \quad \dots(2) \end{aligned}$$

\therefore From (1) and (2)

$$\iint_S (\nabla \times \bar{F} \cdot \hat{n} \, ds = 8\pi$$

Q.6(a) Find the directional derivative of $\phi = xy^2 + yz^2$ at $(2, -1, 1)$ along the line $2(x - 2) = y + 1 = z - 1$. [6]

(A)

$$\phi = xy^2 + yz^2$$

$$\therefore \nabla\phi = iy^2 + j(2xy + z^2) + k(2yz)$$

$$\nabla\phi|_{(2,-1,1)} = i(1) + j(-4 + 1) + k(-2)$$

$$\nabla\phi|_{(2,-1,1)} = 1i - 3j - 2k \quad \dots (1)$$

By given equation of line is $2(x-2) = (y+1) = z-1$

$$\text{i.e. } \frac{x-2}{1} = \frac{y-(-1)}{2} = \frac{z-1}{2} \quad \dots (2)$$

$$\therefore \bar{a} = 1i + 2j + 2k \text{ be any vector along line} \quad \dots (3)$$

$$|\bar{a}| = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$\therefore \hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{1}{3}(1i + 2j + 2k)$$

$$\therefore \left. \begin{array}{l} \text{Directional derivative of} \\ \phi \text{ at } (2, -1, 1) \text{ along the} \\ \text{line } 2(x-2) = y+1 = z-1 \end{array} \right\} = \nabla\phi \cdot \hat{a} = (i - 3j - 2k) \cdot \frac{1}{3}(i + 2j + 2k) = \frac{1}{3}(1 - 6 - 4) = \frac{-9}{3} = -3$$

Q.6(b) Obtain the complex form of Fourier series for the function $f(x) = e^{4x}$ in $0 < x < 4$. [6]

(A)

We know complex form of Fourier series in c to $c + 2l$ is

$$f(x) = \sum_{n=-\infty}^{\infty} A_n e^{i\left(\frac{n\pi x}{l}\right)} \text{ where } A_n = \frac{1}{2l} \int_{x=c}^{c+2l} f(x) e^{-i\left(\frac{n\pi x}{l}\right)} dx$$

On equating the interval $(0, 4)$ with $(c, c + 2l)$

$$c = 0, c + 2l = 4, l = 2 \therefore l = 2$$

\therefore Complex form of Fourier series $(0, 4)$ is

$$f(x) = \sum_{n=-\infty}^{\infty} A_n e^{i\left(\frac{n\pi x}{2}\right)} \quad \dots (1) \quad \text{where}$$

$$A_n = \frac{1}{2 \cdot 2} \int_{x=0}^4 f(x) e^{-i\left(\frac{n\pi x}{2}\right)} dx = \frac{1}{4} \int_{x=0}^4 e^{4x} e^{-i\left(\frac{n\pi x}{2}\right)} dx$$

$$\therefore A_n = \frac{1}{4} \int_{x=0}^4 e^{(8-in\pi)\frac{x}{2}} dx = \frac{1}{4} \left. \frac{e^{(8-in\pi)\frac{x}{2}}}{(8-in\pi)\frac{1}{2}} \right|_{x=0}^4$$

$$\begin{aligned}
 &= \frac{1}{2} \frac{8 + in\pi}{(8 - in\pi)(8 + in\pi)} \left[e^{2(8 - in\pi)} - e^0 \right] \\
 &= \frac{1}{2} \frac{8 + in\pi}{64 + n^2\pi^2} \left[e^{16} e^{-2in\pi} - 1 \right] = \frac{8 + in\pi}{2(64 + n^2\pi^2)} (e^{16} - 1) \quad [\because e^{\pm in\pi} (-1)^n] \\
 &= \frac{e^4 (8 + in\pi)}{64 + n^2\pi^2} \left(\frac{e^4 - e^{-4}}{2} \right) = \frac{e^4 (8 + in\pi)}{64 + 4^2\pi^2} \sinh(4) \quad \dots (2)
 \end{aligned}$$

From (1) and (2)

$$\begin{aligned}
 F(x) &= \sum_{n=-\infty}^{\infty} \frac{e^4 (8 + in\pi) \sinh(4)}{64 + n^2\pi^2} e^{i \left(\frac{n\pi x}{2} \right)} \\
 \therefore f(x) &= e^4 \sinh(4) \sum_{n=-\infty}^{\infty} \left[\frac{8 + in\pi}{64 + n^2\pi^2} \right] e^{i \left[\frac{n\pi x}{2} \right]}
 \end{aligned}$$

Q.6(c) Find half range sine series of the function

[8]

$f(x) = x(3 - x)$ in $0 \leq x \leq 3$

Hence prove that

$$\text{(i) } \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \dots = \frac{\pi^6}{960} \qquad \text{(ii) } \sum_{n=1}^{\infty} \frac{1}{(n)^6} = \frac{\pi^6}{945}$$

(A) We know half range sine series in $(0, 3)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{(n\pi x)}{3} \qquad \dots(1) \quad \text{where}$$

$$b_n = \frac{2}{3} \int_{x=0}^3 f(x) \sin \left(\frac{n\pi x}{3} \right) dx = \frac{2}{3} \int_{x=0}^3 (3x - x^2) \sin \left(\frac{n\pi x}{3} \right) dx$$

$$b_n = \frac{2}{3} \left[(3x - x^2) \left[\frac{-\cos(n\pi x / 9)}{n\pi / 3} \right] - (3 - 2x) \left[\frac{-\sin(n\pi x / 3)}{(n\pi / 3)^2} \right] + (-2) \frac{\cos(n\pi x / 3)}{(n\pi / 3)} \right]_{x=0}^3$$

$$b_n = \frac{2}{3} \left[\frac{-3}{n\pi} (3x - x^2) \cos \left(\frac{n\pi x}{3} \right) + \frac{9}{n^2\pi^2} (3 - 2x) \sin \left(\frac{n\pi x}{3} \right) - 2 \times \frac{27}{n^3\pi^3} \cos \left(\frac{n\pi x}{3} \right) \right]_{x=0}^3$$

$$b_n = \frac{2}{3} \left[\frac{-3}{n\pi} [0 - 0] + \frac{9}{n^2\pi^2} [0 - 0] - \frac{54}{n^3\pi^3} [(-1)^n - 1] \right]$$

$$b_n = \frac{36(1 - (-1)^n)}{n^3\pi^3} \qquad \dots(2)$$

From (1) and (2)

$$F(x) = \sum_{n=1}^{\infty} \frac{36(1 - (-1)^n)}{n^3\pi^3} \cdot \sin \left(\frac{n\pi x}{3} \right) = \frac{36}{\pi^3} \sum_{n=1}^{\infty} \left[\frac{1 - (-1)^n}{n^3} \right] \sin \left(\frac{n\pi x}{3} \right)$$

Now Parse Val's identity of (1) is

$$\sum_{n=1}^{\infty} (b_n)^2 = \frac{2}{3} \int_{x=0}^3 [f(x)]^2 dx = \frac{2}{3} \int_{x=0}^3 x^2 (9 - \sigma x + x^2) dx$$

$$\sum_{n=1}^{\infty} \frac{36 \times 36}{n^6 \pi^6} [1 + 1 - 2(-1)^n] = \frac{2}{3} \int_{x=0}^3 9x^2 - \sigma x^3 + x^4 dx$$

$$\frac{36 \times 36}{\pi^6} \sum_{n\pi} \frac{1 - (-1)^n}{n^6} = \frac{1}{3} \left[\frac{9x^3}{3} - \frac{\sigma x^4}{4} + \frac{x^5}{5} \right]_{x=0}^3$$

$$\frac{36 \times 36}{\pi^6} \left[\frac{2}{1^6} + \alpha l \frac{2}{3^6} + \alpha l \frac{2}{5^6} \dots \right] = \frac{1}{3} \left[\frac{95}{3} - \frac{1}{2} 3^4 + \frac{1}{5} \cdot 3^5 \right]$$

$$\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} \dots = \frac{\pi^6}{36 \times 36 \times 2} \times \frac{9 \times 9}{3} \left[1 - \frac{3}{2} + \frac{3}{5} \right]$$

$$= \frac{\pi^6}{2 \times 48} \times \frac{10 - 15 + 16}{10} = \frac{\pi^6 \cdot 1}{960}$$

$$\therefore \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} \dots = \frac{\pi^6}{960} \quad \dots(3)$$

$$\text{Let } s = \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \frac{1}{6^6} + \dots \quad \dots(4)$$

$$s = \left(\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \dots \right) + \left(\frac{1}{(2 \times 1)^6} + \frac{1}{(2 \times 2)^6} + \frac{1}{(2 \times 3)^6} + \dots \right)$$

$$s = \frac{\pi^6}{960} + \frac{1}{64} \left[\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} \dots \right]$$

$$s = \frac{\pi^6}{960} + \frac{1}{64} \times 5$$

$$s - \frac{s}{64} = \frac{\pi^6}{960}$$

$$\frac{63}{64} s = \frac{\pi^6}{960}$$

$$\therefore s = \frac{\pi^6}{960} \times \frac{64}{63} = \frac{\pi^6}{15 \times 63} = \frac{\pi^6}{945}$$

$$\therefore \sum_{n\pi} \left(\frac{1}{n^6} \right) = \frac{\pi^6}{945}$$

