

Q.1(a) Find Laplace Transformation of $\frac{\sin(3t)}{t}$ [5]

(A) We know $L\{\sin 3t\} = \frac{3}{s^2 + 3^2}$

$$\therefore L\left[\frac{\sin(3t)}{t}\right] = \int_{s=3}^{\infty} \frac{3}{s^2 + 3^2} ds = \frac{3}{3} \tan^{-1}(s/3) \Big|_{s=3}^{\infty}$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s/3) = \frac{\pi}{2} - \tan^{-1}(s/3)$$

$$\therefore L\left(\frac{\sin(3t)}{t}\right) = \cot^{-1}\left(\frac{s}{3}\right)$$

Q.1(b) Prove that $f(z) = \cosh z$ is analytic and find its derivative. [5]

(A) $f(z) = \cosh(z) = \cosh(x+iy) = \cosh x \cos y + i \sinh x \sin y = u + iv$

$$\therefore u = \cosh x \cos y, \quad v = \sinh x \sin y$$

$$u_x = \sinh x \cos y, \quad u_y = -\cosh x \sin y, \quad v_x = \cosh x \sin y, \quad v_y = \sinh x \cos y$$

$$\therefore u_x = v_y \quad \text{and} \quad u_y = -v_x$$

$\therefore u$ and v satisfies C.R. equations

$\therefore f(z) = u + iv$ is analytic

$$\text{Now } u_x|_{(z,0)} = \sinh z \quad v_x|_{(z,0)} = 0$$

$$f'(z) = u_x + iv_x$$

$$= \sinh x \cos y + i \sinh x \sin y$$

put $x = z, \quad y = 0$

$$f'(z) = \sinh z$$

$$\therefore f'(z)|_{(z,0)} = u_x|_{(z,0)} + iv_x|_{(z,0)}$$

$$= \sinh z + i0 = \sinh z$$

Q.1(c) Find Fourier series for $f(x) = 16 - x^2$ over $(-4, +4)$ [5]

(A) $\therefore f(x)$ is even function in $(-4, 4)$ \therefore its Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{4}\right) \quad \dots (1)$$

$$\text{where } a_0 = \frac{2}{4} \int_{x=0}^4 f(x) dx = \frac{1}{2} \int_{x=0}^4 (16 - x^2) dx = \frac{1}{2} \left[16x - \frac{x^3}{3} \right]_0^4 = \frac{1}{2} \left[64 - \frac{64}{3} \right]$$

$$\frac{32}{3} \times 2 = \frac{64}{3} \quad \dots (2)$$

$$a_n = \frac{2}{4} \int_{x=0}^4 f(x) \cos\left(\frac{n\pi x}{4}\right) dx = \frac{1}{2} \int_{x=0}^4 (16 - x^2) \cos\left(\frac{n\pi x}{4}\right) dx$$

$$a_n = \frac{1}{2} \left[(16 - x^2) \frac{\sin(n\pi x / 4)}{n\pi / 4} - (-2x) \left(\frac{-\cos(n\pi x / 4)}{(n\pi / 4)^2} \right) + (-2) \left(\frac{-\sin(n\pi x / 4)}{(n\pi / 4)^3} \right) \right]_{x=0}^4$$

$$a_n = \frac{1}{2} \left[\frac{4}{n\pi} (16 - x^2) \sin\left(\frac{n\pi x}{4}\right) - \frac{16}{n^2 \pi^2} \times 2 \times x \cos\left(\frac{n\pi x}{4}\right) + \frac{64}{n^3 \pi^3} \times 2 \sin\left(\frac{n\pi x}{4}\right) \right]_{x=0}^4$$

$$a_n = \frac{1}{2} \left[\frac{4}{n\pi} (0 - 0) - \frac{32}{n^2 \pi^2} [4(-1)^n - 0] + \frac{128}{n^3 \pi^3} [0 - 0] \right] = \frac{-1}{2} \frac{32}{n^2 \pi^2} 4(-1)^n$$

$$\therefore a_n = \frac{-64}{n^2 \pi^2} (-1)^n \quad \dots (3)$$

From (1), (2), (3)

$$f(x) = \frac{1}{2} \left(\frac{64}{3} \right) + \sum_{n=1}^{\infty} \frac{-64}{n^2 \pi^2} (-1)^n \cos\left(\frac{n\pi x}{4}\right)$$

$$\therefore f(x) = \frac{32}{2} - \frac{64}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{n\pi x}{4}\right)$$

Q.1(d) Evaluate using Cauchy's Residue Theorem $\int_C z^4 e^{1/z} dz$, where $C: |z| = 1$ [5]

(A) $f(z) = z^4 e^{1/z}$

Here pole is $z = 0$, $|0| = 0 < 1$ inside

$$\therefore f(z) = z^4 \left\{ 1 + \frac{1}{z} + \frac{1}{z^2 2!} + \frac{1}{3! z^3} + \frac{1}{4! z^4} + \frac{1}{5! z^5} + \dots \right\}$$

$$f(z) = z^4 + z^3 + \frac{1}{2} z^2 + \frac{1}{6} z + \frac{1}{24} + \frac{1}{120} \frac{1}{z} + \dots$$

Residue at $z = 0$ is coefficient of $\frac{1}{z}$

$$\therefore \text{Residue} = \frac{1}{120}$$

by CR Theorem, $\int_C f(z) dz = 2\pi i$ (sum of residues)

$$\therefore \int_C z^4 e^{1/z} dz = 2\pi i \frac{1}{120} = \frac{\pi i}{60}$$

Q.2(a) Expand $f(z) = \frac{1}{(z-1)(z-2)}$ in (i) $1 < |z-1| < 2$ (ii) $|z| < 1$ [6]

(A) $f(z) = \Rightarrow f(z) = \frac{-1}{z-1} + \frac{1}{z-2}$... (1)

Case (i): $1 < |z-1| < 2$

$$(i) \Rightarrow f(z) = \frac{-1}{z-1} + \frac{1}{(z-1)-1} = \frac{-1}{z-1} + \frac{1}{(z-1)\left[1 - \frac{1}{z-1}\right]}$$

$$= \frac{-1}{z-1} + \frac{1}{(z-1)\left[1 - \frac{1}{z-1}\right]^{-1}}$$

$$f(z) = \frac{-1}{z-1} + \frac{1}{(z-1)} \left\{ 1 + \frac{1}{z-1} + \frac{1}{(z-1)^2} + \dots \right\}$$

Case (ii) $|z| < 1$ i.e. $|z| < 2$ also,

$$(1) \Rightarrow f(z) = \frac{1}{1-z} - \frac{1}{2\left(1 - \frac{z}{2}\right)} = (1-z)^{-1} - \frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1}$$

$$\therefore f(z) = \left[1 + z + z^2 + z^3 + \dots \right] - \frac{1}{2} \left[1 + \frac{z}{2} + \frac{z^2}{4} + \dots \right]$$

Q.2(b) Find the Fourier series for $f(x) = \frac{x-\pi}{4}$; $0 \leq x \leq 2\pi$. [6]

Hence prove that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$

(A) We know Fourier series for the function in $[0, 2\pi]$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) \quad \dots (1)$$

where $a_0 = \frac{1}{\pi} \int_{x=0}^{2\pi} f(x) dx$, $a_n = \frac{1}{\pi} \int_{x=0}^{2\pi} f(x) \cos(nx) dx$, $b_n = \frac{1}{\pi} \int_{x=0}^{2\pi} f(x) \sin(nx) dx$

$$a_0 = \frac{1}{\pi} \int_{x=0}^{2\pi} \frac{1}{4} (x-\pi) dx = \frac{1}{4\pi} \left. \frac{(x-\pi)^2}{2} \right|_0^{2\pi} = \frac{1}{8\pi} [\pi^2 - (-\pi)^2] = 0 \quad \dots (2)$$

$$a_n = \frac{1}{\pi} \int_{x=0}^{2\pi} \left(\frac{x-\pi}{4} \right) \cos(nx) dx = \frac{1}{4\pi} \left[(x-\pi) \left[\frac{\sin(nx)}{n} \right] - 1 \left[\frac{-\cos(nx)}{n^2} \right] \right]_{x=0}^{2\pi}$$

$$\therefore a_n = \frac{1}{4\pi} \left[\frac{1}{n} (x-\pi) \sin(nx) + \frac{1}{n^2} [\cos(nx)] \right]_{x=0}^{2\pi}$$

$$a_n = \frac{1}{4\pi} \left[\frac{1}{n} [0-0] + \frac{1}{n^2} [1-1] \right] = 0 \quad \dots (3)$$

$$b_n = \frac{1}{\pi} \int_{x=0}^{2\pi} \left(\frac{x-\pi}{4} \right) \sin(nx) dx = \frac{1}{4\pi} \left[(x-\pi) \left(\frac{-\cos(nx)}{n} \right) - 1 \left(\frac{-\sin(nx)}{n^2} \right) \right]_{x=0}^{2\pi}$$

$$b_n = \frac{1}{4\pi} \left[-\frac{1}{n} (x-\pi) \cos(nx) + \frac{1}{n^2} \sin(nx) \right]_{x=0}^{2\pi}$$

$$= \frac{1}{4\pi} \left[-\frac{1}{n} [\pi \cdot 1 - (-\pi) \cdot 1] + \frac{1}{n^2} (0-0) \right]$$

$$b_n = \frac{1}{4\pi} \left[-\frac{1}{n} (2\pi) \right] = \frac{-1}{4\pi} \cdot \frac{2\pi}{n} = \frac{-1}{2n} \quad \dots(4)$$

From (1), (2), (3) and (4)

$$f(x) = \frac{1}{2} \cdot 0 + \sum_{n=1}^{\infty} 0 \cdot \cos(nx) + \sum_{n=1}^{\infty} \left(\frac{-1}{2n} \right) \sin(nx)$$

$$\therefore f(x) = -\frac{1}{2} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$$

$$\therefore \frac{1}{4} (x-\pi) = -\frac{1}{2} \sum_{n=1}^{\infty} \left[\frac{\sin(nx)}{n} \right], \quad 0 \leq x \leq 2\pi$$

Put $x = \pi/2$

$$\frac{1}{4} \left(\frac{\pi}{2} - \pi \right) = -\frac{1}{2} \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n}$$

$$\therefore \frac{1}{4} \left(\frac{-\pi}{2} \right) = \frac{1}{2} \left[\frac{1}{1} + 0 - \frac{1}{3} + 0 + \frac{1}{5} + 0 - \frac{1}{7} \dots \right]$$

$$\therefore \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Q.2(c) Find Inverse Laplace transform of

[8]

(i) $\frac{s+19}{(s+9)(s^2+4)}$, (ii) $\frac{e^{-3s}}{(s^2+10s+29)}$

(A) (i) Let $\frac{s+19}{(s+9)(s^2+4)} = \frac{A}{s+9} + \frac{Bs+C}{s^2+4}$... (1)

$\therefore A(s^2+4) + (Bs+C)(s+9) = s+19$... (2)

if $s = -9$, $85A = +10 \therefore A = \frac{2}{17}$

if $s = 0$ $A + 9C = 19$

$$9C = 19 - 4A = 19 - \frac{8}{17} = \frac{315}{17}$$

$$\therefore C = \frac{35}{17}$$

if $s = 1$, $5A + 10(B+C) = 20$

$$B + C = \frac{1}{10} [20 - 5A] = \frac{1}{10} \left[20 - \frac{10}{17} \right] = \left[2 - \frac{1}{17} \right] = \frac{33}{17}$$

$$B = \frac{33}{17} - C = \frac{33}{17} - \frac{35}{17}$$

$$B = \frac{-2}{17}$$

$$\begin{aligned} \therefore \frac{s+19}{(s+9)(s^2+4)} &= \frac{2}{17} \frac{1}{s+9} + \frac{-2}{17} \frac{1}{s^2+4} + \frac{35}{17} \frac{1}{s^2+4} = \frac{2}{17} \cdot \frac{1}{s+9} + \frac{1}{17} \frac{[35-2s]}{s^2+4} \\ &= \frac{1}{17} \left[2 \frac{1}{s+9} + \frac{35}{2} \left(\frac{2}{s^2+2^2} \right) - \frac{2s}{s^2+2^2} \right] \\ \therefore L^{-1} \left[\frac{s+19}{(s+9)(s^2+4)} \right] &= \frac{1}{17} \left[2L^{-1} \left(\frac{1}{s+9} \right) + \frac{35}{2} L^{-1} \left(\frac{2}{s^2+2^2} \right) - 2L^{-1} \left(\frac{s}{s^2+2^2} \right) \right] \\ &= \frac{1}{17} \left[2e^{-9t} + \frac{35}{2} \sin(2t) - 2\cos(2t) \right] \end{aligned}$$

$$(ii) \text{ Let } \bar{f}(s) = \frac{1}{s^2+10s+29} = \frac{1}{s^2+10s+25+4} = \frac{1}{(s+5)^2+2^2}$$

$$\therefore L^{-1}[\bar{f}(s)] = L^{-1} \left[\frac{1}{(s+5)^2+2^2} \right] = e^{-5t} L^{-1} \left(\frac{1}{s^2+2^2} \right) = \frac{e^{-5t}}{2} \sin(2t)$$

$$\begin{aligned} \therefore L^{-1} \left[e^{-3s} \frac{1}{s^2+10s+29} \right] &= L^{-1} [e^{-3s} \bar{f}(s)] = h(t-a) f(t-a) \\ &= h(t-3) \cdot \frac{e^{-5(t-3)}}{2} \sin[2(t-3)] \end{aligned}$$

Q.3(a) Find the analytic function $f(z) = u + iv$

[6]

$$\text{if } u + v = \frac{2 \sin(2x)}{e^{2y} + e^{-2y} - 2 \cos(2x)}$$

$$(A) \quad u + v = \frac{2 \sin(2x)}{2 \cosh(2y) - 2 \cos(2x)} = \frac{\sin(2x)}{\cosh(2y) - \cos(2x)} = (v)$$

$$\text{By given } f(z) = u + iv \quad \therefore i f(z) = iu - v \quad \dots (1)$$

$$\therefore (1+i) f(z) = (u-v) + i(u+v)$$

$$\therefore F(z) = U + iV \quad \dots (2)$$

$$\text{where } F(z) = (1+i) f(z)$$

$$u - v = U \quad \text{and } u + v = V \quad \dots (3)$$

is also analytic

$$\text{Now } F'(z) = U_{x+i} V_x = V_y + iV_x$$

$$\therefore F'(z)|_{(z,0)} = V_y|_{(z,0)} + iV_x|_{(z,0)} \quad \dots (4)$$

$$V_x = \frac{[\cosh(2y) - \cos(2x)]2\cos(2x) - \sin(2x)[2\sin(2x)]}{(\cosh 2y - \cos 2x)^2}$$

$$V_x|_{(z,0)} = \frac{2[\cos 2z(1 - \cos 2z) - \sin^2(2z)]}{(1 - \cos 2z)^2} = \frac{2[\cos(2z) - \cos^2(2z) - \sin^2(2z)]}{(1 - \cos 2z)^2}$$

$$= \frac{-2[1 - \cos(2z)]}{(1 - \cos 2z)^2} = \frac{-2}{1 - \cos 2z} = \frac{-2}{2\sin^2 z} = -\operatorname{cosec}^2 z \quad \dots (5)$$

$$V_y = \frac{-\sin(2xy)}{(\cosh 2y - \cos 2xy)} \times 2\sinh(2y)$$

$$\therefore V_y|_{(z,0)} = \frac{-2\sin(2y)0}{(1 - \cos 2y)^2} = 0 \quad \dots (6)$$

Now from (4), (5) and (6)

$$F'(z)|_{(z,0)} = 0 + i(-\operatorname{cosec}^2 z) = -i\operatorname{cosec}^2(z)$$

$$\therefore F(z) = \int F'(z)|_{(z,0)} dz + c = -i \int \operatorname{cosec}^2 z dz + c = i(+\cot z) + c$$

$$\therefore (1+i)f(z) = +i\cot z + c$$

$$\therefore f(z) = \frac{1}{1+i}i\cot z + \frac{c}{1+i} = \frac{(1-i)i}{2}\cot z + c' = \frac{(1+i)}{2}\cot z + c'$$

Q.3(b) Evaluate $\int_C \frac{e^{2z}}{(z - \pi i)^3} dz$ **where C is** $|z - 2i| = 2$ [6]

(A) Pole $z = \pi i$

$$|\pi i - 2i| = \sqrt{(\pi - 2)^2} < 2 \Rightarrow z = \pi i \text{ lies inside } C$$

$$\therefore \text{by C. I. formula } \int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a)$$

$$\therefore \int_C \frac{e^{2z}}{(z - \pi i)^3} dz = \frac{2\pi i}{2!} f^2(\pi i)$$

$$= \pi i 4e^{2\pi i}$$

$$= 4\pi i [\cos 2\pi + i \sin 2\pi]$$

$$= 4\pi i [1 + i0]$$

$$= 4\pi i$$

$f(z) = e^{2z}$
 $f^2(z) = 4e^{2z}$

Q.3(c) Solve the differential equation $\frac{d^2y}{dt^2} + 4y = f(t)$, **with** $y(0) = 0$ & [8]

$$y'(0) = 1 \text{ and } f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & 1 < t \end{cases}$$

(A) By given $y^{(1)}(t) + 4y(t) = f(t)$

$$\therefore L[y''(t)] + 4Ly(t) = Lf(t) = \int_{t=0}^{\infty} e^{-st} f(t) dt = \int_{t=0}^1 e^{-st} f(t) dt + \int_{t=1}^{\infty} e^{-st} f(t) dt$$

$$s^2 \bar{y}(s) - sy(0) - y'(0) + 4\bar{y}(s) = \int_{t=0}^1 e^{-st} \cdot 1 dt + \int_{t=1}^{\infty} e^{-st} \cdot 0 dt$$

$$(s^2 + 4)\bar{y}(s) - 5 \cdot 0 - 1 = \frac{e^{-st}}{-s} \Big|_{t=0}^1 + 0 = \frac{-1}{s} [e^{-s} - 1] = \frac{1 - e^{-s}}{s}$$

$$\therefore (s^2 + 4)\bar{y}(s) = 1 + \frac{1}{s} - \frac{1}{s} e^{-s}$$

$$\therefore \bar{y}(s) = \frac{1}{s^2 + 4} + \frac{1}{s(s^2 + 4)} - e^{-s} \frac{1}{s(s^2 + 4)}$$

$$\bar{y}(s) = \frac{1}{2} \left(\frac{2}{s^2 + 2^2} \right) + \frac{1}{4} \left[\frac{1}{s} - \frac{s}{s^2 + 2^2} \right] - e^{-s} \frac{1}{4} \left[\frac{1}{s} - \frac{s}{s^2 + 2^2} \right] \quad \dots (1)$$

$$\text{Let } \bar{f}(s) = \frac{1}{4} \left[\frac{1}{s} - \frac{s}{s^2 + 2^2} \right]$$

$$\therefore f(t) = L^{-1} \bar{f}(s) = \frac{1}{4} \left[L^{-1} \left(\frac{1}{s} \right) - L^{-1} \frac{s}{s^2 + 2^2} \right] = \frac{1}{4} [1 - \cos 2t]$$

$$\text{and } L^{-1} [e^{-s} \bar{f}(s)] = H(t - 1) f(t - 1)$$

\therefore from (1)

$$L^{-1}[\bar{y}(s)] = \frac{1}{2} L^{-1} \left(\frac{2}{s^2 + 2^2} \right) + \frac{1}{4} L^{-1} \left(\frac{1}{s} - \frac{s}{s^2 + 2^2} \right) + L^{-1} \left[e^{-s} \frac{1}{4} \left(\frac{1}{s} - \frac{s}{s^2 + 2^2} \right) \right]$$

$$\therefore y(t) = \frac{1}{2} \sin(2t) + \frac{1}{4} (1 - \cos 2t) + H(t - 1) \frac{1}{4} [1 - \cos 2(t - 1)]$$

Q.4(a) Find the orthogonal Trajectory of $3x^2 - 2x^2y + y^2 = \text{constant}$ [6]

(A) $u = 3x^2 - 3y^2 + x^3 - 3xy^2 = \text{constant}$

$$u_x = 6x + 3x^2 - 3y^2, \quad u_y = -6y - 6xy$$

$$dv = V_x dx + V_y dy$$

$$= -u_y dx + u_x dy$$

$$-(-6y - 6xy) dx + (6x + 3x^2 - 3y^2) dy$$

which is exact D.E.

$$\therefore \int dv = \int_{y=\text{constant}} (6y + 6xy) dx + \int_{\text{free from } x} (6x + 3x^2 - 3y^2) dy + c$$

$$v = 6xy + 3x^2y - y^3 + c \text{ is orthogonal trajectory.}$$

Q.4(b) Determine the solution of one dimensional heat equation [6]

$$\frac{\partial y}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \text{ under the boundary conditions } u(0, t) = 0, \quad u(l, t) = 0$$

$$\text{and } u(x, 0) = x, \quad 0 < x < l.$$

(A) Solution of $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ is given by

$$u(x, t) = (C_1 \cos mx + C_2 \sin mx) e^{-m^2 c^2 t} \quad \dots (1)$$

given $u = 0$ when $x = 0$

$$\therefore (1) \Rightarrow 0 = C_1 e^{-m^2 c^2 t} \Rightarrow C_1 = 0$$

$$\therefore (1) \Rightarrow u(x, t) = C_2 \sin(mx) e^{-m^2 c^2 t} \quad \dots (2)$$

given when $x = l$, $u = 0$

$$\therefore (2) \text{ gives } 0 = C_2 \sin(ml) e^{-m^2 c^2 t}$$

$$\Rightarrow \sin(ml) = \sin(n\pi) = 0 \Rightarrow ml = n\pi$$

$$\Rightarrow m = \frac{n\pi}{l}$$

$$\therefore (2) \Rightarrow u(x, t) = C_2 \sin\left(\frac{n\pi}{l}x\right) e^{-\frac{n^2\pi^2}{l^2}c^2 t}$$

adding all above solutions we get the general solution

$$\therefore u(x, t) = \sum b_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2}{l^2}c^2 t} \quad \dots (3)$$

given when $t = 0$, $u = x$

$$\therefore (3) \text{ given } x = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

But this is half range Fourier sine series, in $0 < x < l$

$$\therefore b_n = \frac{1}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx = \frac{2}{l} \int_0^l x \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l} \left\{ \left(x \left[\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}} \right] - (1) \left[\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\frac{n^2\pi^2}{l^2}} \right] \right) \right\}_0^l$$

$$= \frac{2}{l} \left\{ \left[\frac{-l}{n\pi} (-1)^n - 0 \right] - [0 - 0] \right\}$$

$$b_n = -\frac{2l(-1)^n}{n\pi}$$

Q.4(c) Find the Fourier series of the function $f(x) = e^{-x}$, $0 < x < 2\pi$ and [8]

$f(x + 2\pi) = f(x)$. Hence deduce that the value of $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 + 1}$.

$$(A) \quad f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx = \frac{1}{2\pi} \int_0^{2\pi} e^{-x} dx = \frac{-1}{2\pi} [e^{-x}]_0^{2\pi} = \frac{(1 - e^{-2\pi})}{2\pi}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx) dx = \frac{1}{\pi} \int_0^{2\pi} e^{-x} \cos(nx) dx$$

$$= \frac{1}{\pi} \left\{ \frac{e^{-x}}{1+n^2} [-\cos nx + n \sin nx] \right\}_0^{2\pi} = \frac{(1-e^{-2\pi})}{\pi(n^2+1)}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) dx = \frac{1}{\pi} \int_0^{2\pi} e^{-x} [\sin nx] dx$$

$$= \frac{1}{\pi} \left\{ \frac{e^{-x}}{1+n^2} [-\sin nx - n \cos nx] \right\}_0^{2\pi} = \frac{n(1-e^{-2\pi})}{\pi(n^2+1)}$$

$$e^{-x} = \frac{(1-e^{-2\pi})}{2\pi} + \frac{(1-e^{-2\pi})}{\pi} \left\{ \sum_{n=1}^{\infty} \frac{\cos nx}{1+n^2} + \sum_{n=1}^{\infty} \frac{n}{1+n^2} \sin nx \right\}$$

put $x = +\pi$

$$e^{-\pi} = \frac{(1-e^{-2\pi})}{2\pi} + \frac{(1-e^{-2\pi})}{\pi} \left\{ \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} + 0 \right\}$$

$$= \frac{(1-e^{-2\pi})}{2\pi} + \frac{(1-e^{-2\pi})}{\pi} \left\{ -\frac{1}{2} + \sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2} \right\} = \sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2}$$

Q.5(a) Find Inverse Laplace Transform of $\frac{s}{s^4 + 8s^2 + 16}$ using Convolution [6] theorem.

$$(A) \quad \frac{s}{s^4 + 8s^2 + 16} = \frac{s}{(s^2 + 4)^2} = \left(\frac{s}{s^2 + 4} \right) \left(\frac{1}{s^2 + 4} \right) = \bar{f}_1(s) \cdot \bar{f}_2(s) \quad \dots (1)$$

$$\text{where } \bar{f}_1(s) = \frac{s}{s^2 + 4} = \frac{s}{s^2 + 2^2} \text{ and } \bar{f}_2(s) = \frac{1}{s^2 + 4} = \frac{1}{2} \frac{2}{s^2 + 2^2}$$

$$\therefore f_1(t) = \cos(2t) \text{ and } f_2(t) = \frac{1}{2} \sin(2t)$$

\therefore By Convolution Theorem

$$\therefore L^{-1} \left[\frac{s}{s^4 + 8s^2 + 16} \right] = L^{-1} [\bar{f}_1(s) \cdot \bar{f}_2(s)] = \int_{u=0}^t f_1(u) \cdot f_2(t-u) du$$

$$= \int_{u=0}^t \cos(2u) \cdot \frac{1}{2} \sin(2t-2u) \cdot du$$

$$= \frac{1}{2} \int_{u=0}^t \sin(2t-2u) \cos(2u) du$$

$$= \frac{1}{2} \int_{u=0}^t \frac{1}{2} [\sin(2t) + \sin(2t-4u)] du$$

$$= \frac{1}{4} \left[\sin(2t)u - \frac{\cos(2t-4u)}{-4} \right]_{u=0}^t$$

$$\begin{aligned}
 &= \frac{1}{4} \left[\sin(2t)[t-0] + \frac{1}{4} [\cos(-2t) - \cos(2t)] \right] \\
 &= \frac{1}{4} \left[t \sin(2t) + \frac{1}{4} [\cos(2t) - \cos(2t)] \right] \\
 \therefore L^{-1} \left[\frac{s}{s^4 + 8s^2 + 16} \right] &= \frac{t}{4} \sin(2t)
 \end{aligned}$$

Q.5(b) Find the Bilinear Transform which transform the points $z = 2, i, -2$ of z -plane into the points $w = 1, i, -1$ of the w -plane respectively. Also find fixed points of this transformation. [6]

(A) $z_1 = 2, z_2 = i, z_3 = -2$ and $w_1 = 1, w_2 = i$ and $w_3 = -1$
we know Bilinear Transform is

$$\begin{aligned}
 \frac{w-w_1}{w_1-w_2} \cdot \frac{w_2-w_3}{w_3-w} &= \frac{z-z_1}{z_1-z_2} \cdot \frac{z_2-z_3}{z_3-z} \\
 \therefore \frac{w-1}{1-i} \cdot \frac{i+1}{-1-w} &= \frac{z-2}{z-i} \cdot \frac{i+2}{-2-z} \\
 \therefore \frac{w-1}{1+w} \cdot \frac{1+i}{-1+i} &= \frac{z-2}{z+2} \cdot \frac{2+i}{-2+i} \\
 \therefore \frac{w-1}{1+w} &= \frac{z-2}{z+2} \cdot \frac{2+i}{-2+i} \times \frac{-1+i}{1+i} = \frac{(z-2)}{(z+2)} \cdot \frac{(4-3i)}{5} \\
 \frac{w-1}{w+1} &= \frac{4z-8-3iz+6i}{5z+10}
 \end{aligned}$$

By compandant dividendo

$$\begin{aligned}
 \frac{w-1+w+1}{w-1-(w+1)} &= \frac{4z-8-3iz+6i+5z+10}{4z-8-3iz+6i-5z-10} \\
 \frac{2z}{-2} &= \frac{9z+2-3iz+6i}{-z-18-3iz+6i} = \frac{(9-3i)z+(2+6i)}{-(1+3i)z-(18-6i)} \\
 \therefore w &= \frac{(9+3i)z+(2+6i)}{(1+3i)z-(18-6i)} = \frac{3z(3-i)+(6i-2i^2)}{z(1+3i)+(6i-18)} \\
 w &= \frac{3z(3-i)+2i(3-i)}{z(1+3i)+(6i+18i^2)} = \frac{(3-i)(3z+2i)}{z(1+3i)+6i(1+3i)} \\
 w &= \frac{(3-i)(3z+2i)}{(1+3i)(z+6i)} = \frac{(3-i)(3z+2i)}{(-i^2+3i)(z+6i)} = \frac{(3-i)(3z+2i)}{i(z+6i)(3-i)} \\
 \therefore w &= \frac{3z+2i}{iz+6}
 \end{aligned}$$

Q.5(c) Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{x^6 + 1} dx$

[8]

(A) Consider $\int_C \frac{z^2}{z^6 + 1} dz$ where C is large semi-circle as shown in figure

$$\therefore \int_C \frac{z^2}{z^6 + 1} dz = \int_{-R}^R \frac{z^2}{z^6 + 1} dz + \int_{C_1} \frac{z^2}{z^6 + 1} dz \quad \dots (1)$$

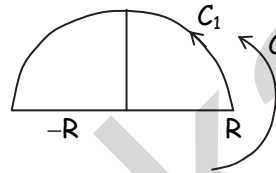
taking $\lim_{R \rightarrow \infty}$, along real axis $y = 0$

$$\therefore z = x + iy \Rightarrow z = x \quad dz = dx$$

$$\text{and } \int_{C_1} \frac{z^2}{z^6 + 1} dz = 0$$

\therefore equation (1) gives,

$$\int_{-\infty}^{\infty} \frac{x^2}{x^6 + 1} dx = \int_C \frac{z^2}{z^6 + 1} dz \quad \dots (2)$$



Here poles are $z^6 = -1 = e^{i\pi} = e^{i(\pi+2n\pi)} \Rightarrow z = e^{i(1+2n)\frac{\pi}{6}}$
 $n = 0, 1, 2, 3, 4, 5$

$$z_0 = e^{i\frac{\pi}{6}}, z_1 = e^{i\frac{3\pi}{6}}, z_2 = e^{i\frac{5\pi}{6}} \text{ are inside and } f(z) = \frac{z^2}{z^6 + 1}$$

$$\text{at } z = z_0 \quad \text{Res.} = \left. \frac{z^2}{6z^5} \right|_{z=z_0} = \frac{1}{6} (z_0)^{-3}$$

$$\text{Similarly at } z = z_1, \text{ Res} = \frac{1}{6} (z_1)^{-3}, \text{ at } z = z_2, \text{ Res} = \frac{1}{6} (z_2)^{-3}$$

using C.R. Theorem for equation (2) we get

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{x^2}{x^6 + 1} dx &= 2\pi i \left\{ \frac{1}{6} (z_0)^{-3} + \frac{1}{6} (z_1)^{-3} + \frac{1}{6} (z_2)^{-3} \right\} \\ &= \frac{\pi i}{3} \left\{ \left(e^{i\frac{\pi}{6}} \right)^{-3} + \left(e^{i\frac{3\pi}{6}} \right)^{-3} + \left(e^{i\frac{5\pi}{6}} \right)^{-3} \right\} \\ &= \frac{\pi i}{3} \left\{ \underset{0}{\cos} \frac{\pi}{2} - i \underset{-1}{\sin} \frac{\pi}{2} + \underset{0}{\cos} \frac{3\pi}{2} - i \underset{-1}{\sin} \frac{3\pi}{2} + \underset{0}{\cos} \frac{5\pi}{2} - i \underset{-1}{\sin} \frac{5\pi}{2} \right\} \\ &= \frac{\pi i}{3} \{ -i - i(-1) - i(1) \} = \frac{-\pi i^2}{3} = \frac{\pi}{3} \end{aligned}$$

Q.6(a) Evaluate $\int_C (x^2 - 2ixy) dz$ along $y = 2x^2$ From $z = 0$ to $z = 3 + 18i$ [6]

(A) $I = \int_C (x^2 - 2ixy)(dx + idy)$;

given $y = 2x^2$

$$\begin{aligned}
 dy &= 4x dx \\
 &= \int_{0+0i}^{3+18i} [x^2 - 2ix \cdot 2x^2] [dx + i4x dx] = \int_0^3 [x^2 - 4ix^3 + 4ix^3 + 16x^4] dx \\
 &= \int_0^3 (x^2 + 16x^4) dx = \left\{ \frac{x^3}{3} + \frac{16x^5}{5} \right\}_0^3 = 9 + 16 \left(\frac{243}{5} \right) - 0 \\
 \therefore \int_C (x^2 - 2ixy) dz &= \frac{3933}{5}
 \end{aligned}$$

Q.6(b) Obtain the complex form of Fourier series for the function [6]

$$f(x) = e^{4x} \text{ in } 0 < x < 4$$

(A) We know complex form of Fourier series in c to $c + 2l$ is

$$f(x) = \sum_{n=-\infty}^{\infty} A_n e^{i \left(\frac{n\pi x}{l} \right)} \text{ where } A_n = \frac{1}{2l} \int_{x=c}^{c+2l} f(x) e^{-i \left(\frac{n\pi x}{l} \right)} dx$$

On equating the interval $(0, 4)$ with $(c, c + 2l)$

$$c = 0, c + 2l = 4, u = 4 \therefore l = 2$$

\therefore Complex form of Fourier series $(0, 4)$ is

$$f(x) = \sum_{n=-\infty}^{\infty} A_n e^{i \left(\frac{n\pi x}{2} \right)} \quad \dots (1)$$

$$\text{where } A_n = \frac{1}{2 \cdot 2} \int_{x=0}^4 f(x) e^{-i \left(\frac{n\pi x}{2} \right)} dx = \frac{1}{4} \int_{x=0}^4 e^{4x} e^{-i \left(\frac{n\pi x}{2} \right)} dx$$

$$\begin{aligned}
 \therefore A_n &= \frac{1}{4} \int_{x=0}^4 e^{(8-in\pi)\frac{x}{2}} dx = \frac{1}{4} \left. \frac{e^{(8-in\pi)\frac{x}{2}}}{(8-in\pi)\frac{1}{2}} \right|_{x=0}^4 \\
 &= \frac{1}{2} \frac{8+in\pi}{(8-in\pi)(8+in\pi)} [e^{2(8-in\pi)} - e^0] \\
 &= \frac{1}{2} \frac{8+in\pi}{64+n^2\pi^2} [e^{16} e^{-2in\pi} - 1] = \frac{8+in\pi}{2(64+n^2\pi^2)} (e^{16} - 1) \quad [\because e^{\pm in\pi} (-1)^n] \\
 &= \frac{e^4 (8+in\pi)}{64+n^2\pi^2} \left(\frac{e^4 - e^{-4}}{2} \right) = \frac{e^4 (8+in\pi)}{64+4^2\pi^2} \sinh(4) \quad \dots (2)
 \end{aligned}$$

From (1) and (2)

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{e^4 (8+in\pi) \sinh(4)}{64+n^2\pi^2} e^{i \left(\frac{n\pi x}{2} \right)}$$

$$\therefore f(x) = e^4 \sinh(4) \sum_{n=-\infty}^{\infty} \left[\frac{8+in\pi}{64+n^2\pi^2} \right] e^{i \left[\frac{n\pi x}{2} \right]}$$

Q.6(c) Find half range sine series of the function

[8]

$$f(x) = x(3-x) \text{ in } 0 \leq x \leq 3$$

Hence prove that

$$(i) \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \dots = \frac{\pi^6}{960}$$

$$(ii) \sum_{n=1}^{\infty} \frac{1}{(n)^6} = \frac{\pi^6}{945}$$

(A) We know half range sine series in (0, 3) is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{(n\pi x)}{3} \quad \dots(1)$$

$$\text{where } b_n = \frac{2}{3} \int_{x=0}^3 f(x) \sin \left(\frac{n\pi x}{3} \right) dx = \frac{2}{3} \int_{x=0}^3 (3x - x^2) \sin \left(\frac{n\pi x}{3} \right) dx$$

$$b_n = \frac{2}{3} \left[(3x - x^2) \left[\frac{-\cos(n\pi x/3)}{n\pi/3} \right] - (3 - 2x) \left[\frac{-\sin(n\pi x/3)}{(n\pi/3)^2} \right] + (-2) \frac{\cos(n\pi x/3)}{(n\pi/3)} \right]_{x=0}^3$$

$$b_n = \frac{2}{3} \left[\frac{-3}{n\pi} (3x - x^2) \cos \left(\frac{n\pi x}{3} \right) + \frac{9}{n^2 \pi^2} (3 - 2x) \sin \left(\frac{n\pi x}{3} \right) - 2 \times \frac{27}{n^3 \pi^3} \cos \left(\frac{n\pi x}{3} \right) \right]_{x=0}^3$$

$$b_n = \frac{2}{3} \left[\frac{-3}{n\pi} [0 - 0] + \frac{9}{n^2 \pi^2} [0 - 0] - \frac{54}{n^3 \pi^3} [(-1)^n - 1] \right]$$

$$b_n = \frac{36(1 - (-1)^n)}{n^3 \pi^3} \quad \dots(2)$$

From (1) and (2)

$$F(x) = \sum_{n=1}^{\infty} \frac{36(1 - (-1)^n)}{n^3 \pi^3} \cdot \sin \left(\frac{n\pi x}{3} \right) = \frac{36}{\pi^3} \sum_{n=1}^{\infty} \left[\frac{1 - (-1)^n}{n^3} \right] \sin \left(\frac{n\pi x}{3} \right)$$

Now Parse Val's identity of (1) is

$$\sum_{n=1}^{\infty} (b_n)^2 = \frac{2}{3} \int_{x=0}^3 [f(x)]^2 dx = \frac{2}{3} \int_{x=0}^3 x^2 (9 - 6x + x^2) dx$$

$$\sum_{n=1}^{\infty} \frac{36 \times 36}{n^6 \pi^6} [1 + 1 - 2(-1)^n] = \frac{2}{3} \int_{x=0}^3 9x^2 - 6x^3 + x^4 dx$$

$$\frac{36 \times 36}{\pi^6} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^6} = \frac{1}{3} \left[9x^3 - \frac{6x^4}{4} + \frac{x^5}{5} \right]_{x=0}^3$$

$$\frac{36 \times 36}{\pi^6} \left[\frac{2}{1^6} + a \frac{2}{3^6} + a \frac{2}{5^6} \dots \right] = \frac{1}{3} \left[\frac{95}{3} - \frac{1}{2} 3^4 + \frac{1}{5} \cdot 35 \right]$$

$$\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} \dots = \frac{\pi^6}{36 \times 36 \times 2} \times \frac{9 \times 9}{3} \left[1 - \frac{3}{2} + \frac{3}{5} \right]$$

$$= \frac{\pi^6}{2 \times 48} \times \frac{10 - 15 + 16}{10} = \frac{\pi^6 \cdot 1}{960}$$

$$\therefore \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} \dots = \frac{\pi^6}{960} \quad \dots(3)$$

$$\text{Let } s = \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \frac{1}{6^6} + \dots \quad \dots(4)$$

$$s = \left(\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \dots \right) + \left(\frac{1}{(2 \times 1)^6} + \frac{1}{(2 \times 2)^6} + \frac{1}{(2 \times 3)^6} + \dots \right)$$

$$s = \frac{\pi^6}{960} + \frac{1}{64} \left[\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} \dots \right]$$

$$s = \frac{\pi^6}{960} + \frac{1}{64} \times 5$$

$$s - \frac{s}{64} = \frac{\pi^6}{960}$$

$$\frac{63}{64} s = \frac{\pi^6}{960}$$

$$\therefore s = \frac{\pi^6}{960} \times \frac{64}{63} = \frac{\pi^6}{15 \times 63} = \frac{\pi^6}{945}$$

$$\therefore \sum_{n=1}^{\infty} \left(\frac{1}{n^6} \right) = \frac{\pi^6}{945}$$

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