Applied Mathematics-III

Time : 3 Hrs. Prelim Question Paper Solution **[Marks : 80**

Q.1(a) Find Laplace Transformation of $\frac{\sin(3t)}{t}$ **[5]**

(A) We know L {sin3t} $= \frac{3}{s^2 + 3^2}$

$$\therefore L\left[\frac{\sin(3t)}{t}\right] = \int_{s=3}^{\infty} \frac{3}{s^2+3^2} ds = \frac{3}{3}\tan^{-1}(s/3)\Big|_{s=3}^{\infty}$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s/3) = \frac{\pi}{2} - \tan^{-1}(s/3)$$

$$\therefore L\left(\frac{\sin(3t)}{t}\right) = \cot^{-1}\left(\frac{s}{3}\right)$$

Q.1(b) Prove that f(z) = cosh z is analytic and find its derivative. **[5]**

(A) f(z) = cosh(z) = cosh(x+iy) = coshx cosy + i sinhx siny = u + iv

$\therefore$ u = coshx cosy, v = sinhx siny

u_x = sinhx cosy, u_y = -coshx siny, v_x = coshx siny, v_y = sinhx cosy

$\therefore$ $u_x = v_y$ and $u_y = - v_x$

$\therefore$ u and V satisfies C.R. equations

$\therefore$ f(z) = u + iv is analytic

Now $ux|_{(z,0)}$ = sinhz $vx|_{(z,0)}$ = 0

f'(z) $= u_x + iv_x$

= sinhx cosy + i sinhx siny

put x = z, y = 0

f'(z) = sinhz

$\therefore$ $f'(z)|_{(z,0)} = ux|_{(z,0)} + ivx|_{(z,0)}$

= sinhz + i0 = sinhz

Q.1(c) Find Fourier series for f(x) = 16 − x^2 over (−4, +4) **[5]**

(A) $\because$ f(x) is even function in (-4, 4) $\therefore$ its Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} an\cos\left(\frac{n\pi x}{4}\right) \quad \ldots (1)$$

where $a_0 = \frac{2}{4}\int_{x=0}^{4} f(x)\,dx = \frac{1}{2}\int_{x=0}^{4}(16-x^2)\,dx = \frac{1}{2}\left[16x - \frac{x^3}{3}\right]_0^4 = \frac{1}{2}\left[64 - \frac{64}{3}\right]$

$$\frac{32}{3}\times 2 = \frac{64}{3} \qquad \text{... (2)}$$

$$a_n = \frac{2}{4}\int_{x=0}^{4} f(x)\cos\left(\frac{n\pi x}{4}\right)dx = \frac{1}{2}\int_{x=0}^{4}\left(16-x^2\right)\cos\left(\frac{n\pi x}{4}\right)dx$$

$$a_n = \frac{1}{2}\left[\left(16-x^2\right)\frac{\sin(n\pi x/4)}{n\pi/4} - (-2x)\left(\frac{-\cos(n\pi x/4)}{(n\pi/4)^2}\right) + (-2)\left(\frac{-\sin(n\pi x/4)}{(n\pi/4)^3}\right)\right]_{x=0}^{4}$$

$$a_n = \frac{1}{2}\left[\frac{4}{n\pi}\left(16-x^2\right)\sin\left(\frac{n\pi x}{4}\right) - \frac{16}{n^2\pi^2}\times 2\ x\cos\left(\frac{n\pi x}{4}\right) + \frac{64}{n^3\pi^3}\times 2\sin\left(\frac{n\pi x}{4}\right)\right]_{x=0}^{4}$$

$$a_n = \frac{1}{2}\left[\frac{4}{n\pi}(0-0) - \frac{32}{n^2\pi^2}\left[4(-1)^n - 0\right] + \frac{128}{n^3\pi^3}\left[0-0\right]\right] = \frac{-1}{2}\frac{32}{n^2\pi^2}4(-1)^n$$

$$\therefore a_n = \frac{-64}{n^2\pi^2}(-1)^n \qquad \text{... (3)}$$

From (1), (2), (3)

$$f(x) = \frac{1}{2}\left(\frac{64}{3}\right) + \sum_{n=1}^{\infty}\frac{-64}{n^2\pi^2}(-1)^n\cos\left(\frac{n\pi x}{4}\right)$$

$$\therefore \quad f(x) = \frac{32}{2} - \frac{64}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{4}\right)$$

Q.1(d) Evaluate using Cauchy's Residue Theorem $\int_C z^4 e^{1/z}dz$, where C:|z| = 1 [5]

(A) $f(z) = z^4 e^{1/z}$

Here pole is z = 0, |0| = 0 < 1 inside

$$\therefore \quad f(z) = z^4\left\{1 + \frac{1}{z} + \frac{1}{z^2 2!} + \frac{1}{3!z^3} + \frac{1}{4!z^4} + \frac{1}{5!z^5} + ...\right\}$$

$$f(z) = z^4 + z^3 + \frac{1}{2}z^2 + \frac{1}{6}z + \frac{1}{24} + \frac{1}{120}\frac{1}{z} + ...$$

Residue at z = 0 is coefficient of $\frac{1}{z}$

$$\therefore \quad \text{Residue} = \frac{1}{120}$$

by CR Theorem, $\int_c f(z)dz = 2\pi i$ (sum of residues)

$$\therefore \quad \int_c z^4 e^{1/z}dz = 2\pi i\ \frac{1}{120} = \frac{\pi i}{60}$$

Q.2(a) Expand $f(z) = \dfrac{1}{(z-1)(z-2)}$ in (i) $1 < |z-1| < 2$ (ii) $|z| < 1$ **[6]**

(A) $f(z) = \Rightarrow f(z) = \dfrac{-1}{z-1} + \dfrac{1}{z-2}$ ... (1)

Case (i) : $1 < |z-1| < 2$

$$\text{(i)} \Rightarrow f(z) = \frac{-1}{z-1} + \frac{1}{(z-1)-1} = \frac{-1}{z-1} + \frac{1}{(z-1)\left[1-\frac{1}{z-1}\right]}$$

$$= \frac{-1}{z-1} + \frac{1}{(z-1)}\left[1-\frac{1}{z-1}\right]^{-1}$$

$$f(z) = \frac{-1}{z-1} + \frac{1}{(z-1)}\left\{1+\frac{1}{z-1}+\frac{1}{(z-1)^2}+\ldots\right\}$$

Case (ii) $|z| < 1$ i.e. $|z| < 2$ also,

$$(1) \Rightarrow f(z) = \frac{1}{1-z} - \frac{1}{2\left(1-\frac{z}{2}\right)} = (1-z)^{-1} - \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$\therefore \quad f(z) = \left[1+z+z^2+z^3+\ldots\right] - \frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+\ldots\right]$$

Q.2(b) Find the Fourier series for $f(x) = \dfrac{x-\pi}{4}$; $0 \le x \le 2\pi$. **[6]**

Hence prove that $1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots = \dfrac{\pi}{4}$

(A) We know Fourier series for the function in $[0, 2\pi]$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_v \sin(nx) \quad \ldots (1)$$

where $a_0 = \dfrac{1}{\pi}\displaystyle\int_{x=0}^{2\pi} f(x)dx$, $a_n = \dfrac{1}{\pi}\displaystyle\int_{x=0}^{2\pi} f(x)\cos(nx)dx$, $b_n = \dfrac{1}{\pi}\displaystyle\int_{x=0}^{2\pi} f(x)\sin(nx)dx$

$$a_0 = \frac{1}{\pi}\int_{x=0}^{2\pi} \frac{1}{4}(x-\pi)dx = \frac{1}{4\pi}\left.\frac{(x-\pi)^2}{2}\right|_0^{2\pi} = \frac{1}{8\pi}[\pi^2 - (-\pi)^2] = 0 \quad \ldots (2)$$

$$a_n = \frac{1}{\pi}\int_{x=0}^{2\pi}\left(\frac{x-\pi}{4}\right)\cos(nx)dx = \frac{1}{4\pi}\left[(x-\pi)\left[\frac{\sin(nx)}{n}\right] - 1\left[\frac{-\cos(nx)}{n^2}\right]\right]_{x=0}^{2\pi}$$

$$\therefore \quad a_n = \frac{1}{4\pi}\left[\frac{1}{n}(x-\pi)\sin(nx) + \frac{1}{n^2}[\cos(nx)]\right]_{x=0}^{2\pi}$$

$$a_n = \frac{1}{4\pi}\left[\frac{1}{n}[0-0] + \frac{1}{n^2}[1-1]\right] = 0 \quad \ldots (3)$$

$$b_n = \frac{1}{\pi}\int_{x=0}^{2\pi}\left(\frac{x-\pi}{4}\right)\sin(nx)\,dx = \frac{1}{4\pi}\left[(x-\pi)\left(\frac{-\cos(nx)}{n}\right)-1\left(\frac{-\sin(nx)}{n^2}\right)\right]_{x=0}^{2\pi}$$

$$b_n = \frac{1}{4\pi}\left[-\frac{1}{n}(x-\pi)\cos(nx)+\frac{1}{n^2}\sin(nx)\right]_{x=0}^{2\pi}$$

$$= \frac{1}{4\pi}\left[-\frac{1}{n}\left[\pi\cdot 1-(-\pi)\cdot 1\right]+\frac{1}{n^2}(0-0)\right]$$

$$b_n = \frac{1}{4\pi}\left[-\frac{1}{n}(2\pi)\right] = \frac{-1}{4\pi}\cdot\frac{2\pi}{n} = \frac{-1}{2n} \quad \ldots(4)$$

From (1), (2), (3) and (4)

$$f(x) = \frac{1}{2}\cdot 0+\sum_{n=1}^{\infty}0\cdot\cos(nx)+\sum_{n=1}^{\infty}\left(\frac{-1}{2n}\right)\sin(nx)$$

$$\therefore \quad f(x) = -\frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}$$

$$\therefore \quad \frac{1}{4}(x-\pi) = -\frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{\sin(nx)}{n}\right], \qquad 0 \le x \le 2\pi$$

Put $x = \pi/2$

$$\frac{1}{4}\left(\frac{\pi}{2}-\pi\right) = \frac{-1}{2}\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{n}$$

$$\therefore \quad \frac{1}{4}\left(\frac{-\pi}{2}\right) = \frac{1}{2}\left[\frac{1}{1}+0-\frac{1}{3}+0+\frac{1}{5}+0-\frac{1}{7}\ldots\right]$$

$$\therefore \quad \frac{\pi}{4} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$$

Q.2(c) Find Inverse Laplace transform of **[8]**

(i) $\dfrac{s+19}{(s+9)(s^2+4)}$, **(ii)** $\dfrac{e^{-3s}}{(s^2+10s+29)}$.

(A) (i) Let $\dfrac{s+19}{(s+9)(s^2+4)} = \dfrac{A}{s+9}+\dfrac{Bs+C}{s^2+4}$ ... (1)

$\therefore$ $A(s^2 + 4) + (Bs + C)(s + 9) = s + 19$ ... (2)

if $s = -9$, $85A = +10$ $\therefore A = \dfrac{2}{17}$

if $s = 04$ $A + 9C = 19$

$$9C = 19 - 4A = 19-\frac{8}{17} = \frac{315}{17}$$

$$\therefore \quad C = \frac{35}{17}$$

if $s = 1$, $5A + 10(B + C) = 20$

$$B + C = \frac{1}{10}\left[20 - 5A\right] = \frac{1}{10}\left[20 - \frac{10}{17}\right] = \left[2 - \frac{1}{17}\right] = \frac{33}{17}$$

$$B = \frac{33}{17} - C = \frac{33}{17} - \frac{35}{17}$$

$$B = \frac{-2}{17}$$

$$\therefore \frac{s+19}{(s+9)(s^2+4)} = \frac{2}{17}\frac{1}{s+9} + \frac{\frac{-2}{17}s + \frac{35}{17}}{s^2+4} = \frac{2}{17}\cdot\frac{1}{s+9} + \frac{1}{17}\frac{\left[35 - 2s\right]}{s^2+4}$$

$$= \frac{1}{17}\left[2\frac{1}{s+g} + \frac{35}{2}\left(\frac{2}{s^2+2^2}\right) - \frac{2s}{s^2+2^2}\right]$$

$$\therefore L^{-1}\left[\frac{s+19}{(s+9)(s^2+4)}\right] = \frac{1}{17}\left[2L^{-1}\left(\frac{1}{s+9}\right) + \frac{35}{2}L^{-1}\left(\frac{2}{s^2+2^2}\right) - 2L^{-1}\left(\frac{s}{s^2+2^2}\right)\right]$$

$$= \frac{1}{17}\left[2e^{-9t} + \frac{35}{2}\sin(2t) - 2\cos(2t)\right]$$

(ii) Let $\bar{f}(s) = \frac{1}{s^2+10s+29} = \frac{1}{s^2+10s+25+4} = \frac{1}{(s+5)^2+2^2}$

$$\therefore L^{-1}\left[\bar{f}(s)\right] = L^{-1}\left[\frac{1}{(s+5)^2+2^2}\right] = e^{-5t}L^{-1}\left(\frac{1}{s^2+2^2}\right) = \frac{e^{-5t}}{2}\sin(2t)$$

$$\therefore L^{-1}\left[e^{-3s}\frac{1}{s^2+10s+29}\right] = L^{-1}\left[e^{-3s}\,\bar{f}(s)\right] = h(t-a)\,f(t-a)$$

$$= h(t-3)\cdot\frac{e^{-5(t-3)}}{2}\sin\left[2(t-3)\right]$$

Q.3(a) Find the analytic function f(z) = u + iv **[6]**

if $u + v = \dfrac{2\sin(2x)}{e^{2y} + e^{-2y} - 2\cos(2x)}$

(A) $u + v = \dfrac{2\sin(2x)}{2\cosh(2y) - 2\cos(2x)} = \dfrac{\sin(2x)}{\cosh(2y) - \cos(2x)} = (v)$

By given f(z) = u + iv $\quad\therefore$ i f(z) = iu – v ... (1)

$\therefore$ (1 + i) f(z) = (u - v) + i(u + v)

$\therefore$ F(z) = U + iV ... (2)

where F(z) = (1 + i) f(z)

u – v = U and u + v = V ... (3)

is also analytic

Now $F'(z) = U_{xei}\,V_x = V_y + iV_x$

$\therefore F'(z)|_{(z,0)} = V_y|_{(z,0)} + iV_x|_{(z,0)}$ ... (4)

$$V_x = \frac{[\cosh(2y) - \cos(2x)]2\cos(2x) - \sin(2x)[2\sin(2x)]}{(\cosh 2y - \cos 2x)^2}$$

$$V_x|_{(z,\,0)} = \frac{2[\cos 2z(1-\cos 2z) - \sin^2(2z)]}{(1-\cos 2z)^2} = \frac{2[\cos(2z) - \cos^2(2z) - \sin^2(2z)]}{(1-\cos 2z)^2}$$

$$= \frac{-2[1-\cos(2z)]}{(1-\cos 2z)^2} = \frac{-2}{1-\cos 2z} = \frac{-2}{2\sin^2 z} = -\text{cosec}^2 z \qquad \text{... (5)}$$

$$V_y = \frac{-\sin(2xy)}{(\cosh 2y - \cos 2xy)} \times 2\sinh(2y)$$

$$\therefore \quad V_y|_{(z,0)} = \frac{-2\sin(2y)0}{(1-\cos 2y)^2} = 0 \qquad \text{... (6)}$$

Now from (4), (5) and (6)

$$F'(z)|_{(z,\,0)} = 0 + i(-\text{cosec}^2 z) = -i\text{cosec}^2(z)$$

$$\therefore F(z) = \int F'(z)|_{(z,0)}\, dz + c = -i\int \text{cosec}^2 z\, dz + c = i(+\cot z) + c$$

$$\therefore (1 + i)\; f(z) = +i\cot z + c$$

$$\therefore f(z) = \frac{1}{1+i} i\cot z + \frac{c}{1+i} = \frac{(1-i)i}{2}\cot z + c' = \frac{(1+i)}{2}\cot z + c'$$

Q.3(b) Evaluate $\int_c \frac{e^{2z}}{(z - \pi i)^3} dz$ where C is $|z - 2i| = 2$ **[6]**

(A) Pole $z = \pi i$

$|\pi i - 2i| = \sqrt{(\pi - 2)^2} < 2 \Rightarrow z = \pi i$ lies inside C

$\therefore$ by C. I. formula $\int_c \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a)$

$$\therefore \int_c \frac{e^{2z}}{(z-\pi i)^3} dz = \frac{2\pi i}{2!} f^2(\pi i)$$

$f(z) = e^{2z}$

$f^2(z) = 4e^{2z}$

$= \pi i\; 4e^{2\pi i}$

$= 4\pi i\,[\cos 2\pi + i\sin 2\pi]$

$= 4\pi i\,[1 + i0]$

$= 4\pi i$

Q.3(c) Solve the differential equation $\frac{d^2y}{dt^2} + 4y = f(t)$, with $y(0) = 0$ & $y'(0) = 1$ and $f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & 1 < t \end{cases}$ **[8]**

(A) By given $y^{11}(t) + 4y(t) = f(t)$

$$\therefore \quad L[y''(t)] + 4\,Ly(t) = Lf(t) = \int_{t=0}^{\infty} e^{-st} f(t)dt = \int_{t=0}^{1} e^{-st} f(t)dt + \int_{t=1}^{\infty} e^{-st} f(t)dt$$

$$s^2\bar{y}(s) - sy(0) - y'(0) + 4\bar{y}(s) = \int_{t=0}^{1} e^{-st}\cdot 1dt + \int_{t=1}^{\infty} e^{-st}\cdot 0dt$$

$$(s^2+4)\,\bar{y}(s) - 5.0 - 1 = \left.\frac{e^{-st}}{-s}\right|_{t=0}^{1} + 0 = \frac{-1}{s}[e^{-s}-1] = \frac{1-e^{-s}}{s}$$

$$\therefore \quad (s^2+4)\,\bar{y}(s) = 1 + \frac{1}{s} - \frac{1}{s}e^{-s}$$

$$\therefore \quad \bar{y}(s) = \frac{1}{s^2+4} + \frac{1}{s(s^2+4)} - e^{-s}\frac{1}{s(s^2+4)}$$

$$\bar{y}(s) = \frac{1}{2}\left(\frac{2}{s^2+2^2}\right) + \frac{1}{4}\left[\frac{1}{s} - \frac{s}{s^2+2^2}\right] - e^{-s}\frac{1}{4}\left[\frac{1}{s} - \frac{s}{s^2+2^2}\right] \quad \text{... (1)}$$

Let $\bar{f}(s) = \frac{1}{4}\left[\frac{1}{s} - \frac{s}{s^2+2^2}\right]$

$$\therefore \quad f(t) = L^{-1}\bar{f}(s) = \frac{1}{4}\left[L^{-1}\left(\frac{1}{s}\right) - L^{-1}\frac{s}{s^2+2^2}\right] = \frac{1}{4}\left[1-\cos 2t\right]$$

and $L^{-1}[e^{(-s)}\bar{f}(s)] = H(t-1)\ f(t-1)$

$\therefore$ from (1)

$$L^{-1}[\bar{y}(s)] = \frac{1}{2}L^{-1}\left(\frac{2}{s^2+2^2}\right) + \frac{1}{4}L^{-1}\left(\frac{1}{s} - \frac{s}{s^2+2^2}\right) + L^{-1}\left[e^{-s}\frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+2^2}\right)\right]$$

$$\therefore \quad y(t) = \frac{1}{2}\sin(2t) + \frac{1}{4}(1-\cos 2t) + H(t-1)\frac{1}{4}\left[1-\cos 2(t-1)\right]$$

Q.4(a) Find the orthogonal Trajectory of $3x^2 - 2x^2y + y^2$ = constant **[6]**

(A) $u = 3x^2 - 3y^2 + x^3 - 3xy^2$ = constant

$u_x = 6x + 3x^2 - 3y^2$, $u_y = -6y - 6xy$

$dv = V_x dx + V_y dy$

$= -u_y dx + u_x dy$

$-(-6y - 6xy)\,dx + (6x + 3x^2 - 3y^2)\,dy$

which is exact D.E.

$$\therefore \quad \int dv = \int_{y=\text{constant}} (6y+6xy)dx + \int_{\text{free from } x} (6x+3x^2-3y^2)dy + c$$

$v = 6xy + 3x^2y - y^3 + c$ is orthogonal trajectory.

Q.4(b) Determine the solution of one dimensional heat equation $\frac{\partial y}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ under the boundary conditions $u(0, t) = 0$, $u(l, t) = 0$ and $u(x, 0) = x$, $0 < x < l$. **[6]**

(A) Solution of $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ is given by

$$u(x, t) = (C_1\cos mx + C_2\sin mx)\ e^{-m^2c^2t} \quad \text{... (1)}$$

given u = 0 when x = 0

$\therefore$ (1) $\Rightarrow$ $0 = C_1\ e^{-m^2c^2t}$ $\Rightarrow$ $C_1 = 0$

$\therefore$ (1) $\Rightarrow$ $u(x, t) = C_2 \sin(mx)\ e^{-m^2c^2t}$ ... (2)

given when x = l, u = 0

$\therefore$ (2) gives $0 = C_2 \sin(ml)\, e^{-m^2c^2t}$

$\Rightarrow$ $\sin(ml) = \sin(n\pi) = 0$ $\Rightarrow m\,l = n\pi$

$\Rightarrow$ $m = \frac{n\pi}{l}$

$\therefore$ (2) $\Rightarrow$ $u(x, t) = C_2\ \sin\left(\frac{n\pi}{l}x\right)\ e^{-\frac{n^2\pi^2}{l^2}c^2t}$

adding all above solutions we get the general solution

$\therefore$ $u(x, t) = \sum bn \sin\left(\frac{n\pi x}{l}\right) e^{\frac{-n^2\pi^2}{l^2}c^2t}$ ... (3)

given when t = 0, u = x

$\therefore$ (3) given $x = \sum_{n=1}^{\infty} bn \sin\left(\frac{n\pi x}{l}\right)$

But this is half range Fourier sine series, in 0 < x < l

$$\therefore\quad b_n = \frac{1}{\frac{l}{2}}\int_0^l f(x)\sin\left(\frac{nx\pi}{l}\right)dx = \frac{2}{l}\int_0^l x\sin\left(\frac{nx\pi}{l}\right)dx$$

$$= \frac{2}{l}\left\{(x)\left[\frac{-\cos\left(\frac{nx\pi}{l}\right)}{\frac{n\pi}{l}}\right] - (1)\left[\frac{-\sin\left(\frac{nx\pi}{l}\right)}{\frac{n^2\pi^2}{l^2}}\right]\right\}_0^l$$

$$= \frac{2}{l}\left\{\left[\frac{-l}{\frac{n\pi}{l}}(-1)^n - 0\right] - \left[0 - 0\right]\right\}$$

$$b_n = -\frac{2l(-1)^n}{n\pi}$$

Q.4(c) Find the Fourier series of the function $f(x) = e^{-x}$, $0 < x < 2\pi$ and $f(x + 2\pi) = f(x)$. Hence deduce that the value of $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 + 1}$. **[8]**

(A) $f(x) = a_0 + \sum_{n=1}^{\infty}\left[an\cos(nx) + bn\sin(nx)\right]$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} e^{-x}dx = \frac{-1}{2\pi}\left[e^{-x}\right]_0^{2\pi} = \frac{\left(1 - e^{-2\pi}\right)}{2\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left\{\frac{e^{-x}}{1+n^2}\left[-\cos nx + n\sin nx\right]\right\}_0^{2\pi} = \frac{\left(1-e^{-2\pi}\right)}{\pi\left(n^2+1\right)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\left[\sin nx\right]dx$$

$$= \frac{1}{\pi}\left\{\frac{e^{-x}}{1+n^2}\left[-\sin nx - n\cos nx\right]\right\}_0^{2\pi} = \frac{n\left(1-e^{-2\pi}\right)}{\pi\left(n^2+1\right)}$$

$$e^{-x} = \frac{\left(1-e^{-2\pi}\right)}{2\pi} + \frac{\left(1-e^{-2\pi}\right)}{\pi}\left\{\sum_{n=1}^{\infty}\frac{\cos nx}{1+n^2} + \sum_{n=1}^{\infty}\frac{n}{1+n^2}\sin nx\right\}$$

put $x = +\pi$

$$e^{-\pi} = \frac{\left(1-e^{-2\pi}\right)}{2\pi} + \frac{\left(1-e^{-2\pi}\right)}{\pi}\left\{\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2} + 0\right\}$$

$$= \frac{\left(1-e^{-2\pi}\right)}{2\pi} + \frac{\left(1-e^{-2\pi}\right)}{\pi}\left\{-\frac{1}{2} + \sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2}\right\} \qquad = \sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2}$$

Q.5(a) Find Inverse Laplace Transform of $\frac{s}{s^4+8s^2+16}$ using Convolution theorem. **[6]**

(A) $$\frac{s}{s^4+8s^2+16} = \frac{s}{(s^2+4)^2} = \left(\frac{s}{s^2+4}\right)\left(\frac{1}{s^2+4}\right) = \bar{f}_1(s)\cdot\bar{f}_2(s) \qquad \ldots (1)$$

where $\bar{f}_1(s) = \frac{s}{s^2+4} = \frac{s}{s^2+2^2}$ and $\bar{f}_2(s) = \frac{1}{s^2+4} = \frac{1}{2}\frac{2}{s^2+2^2}$

$\therefore$ $f_1(t) = \cos(2t)$ and $f_2(t) = \frac{1}{2}\sin(2t)$

$\therefore$ By Convolution Theorem

$$\therefore L^{-1}\left[\frac{s}{s^4+8s^2+16}\right] = L^{-1}\left[\bar{f}_1(s)\cdot\bar{f}_2(s)\right] = \int_{u=0}^{t} f_1(u)\cdot f_2(t-u)\,du$$

$$= \int_{u=0}^{t}\cos(2u)\cdot\frac{1}{2}\sin(2t-2u)\cdot du$$

$$= \frac{1}{2}\int_{u=0}^{t}\sin(2t-2u)\cos(2u)\,du$$

$$= \frac{1}{2}\int_{u=0}^{t}\frac{1}{2}\left[\sin(2t)+\sin(2t-4u)\right]du$$

$$= \frac{1}{4}\left[\sin(2t)u - \frac{\cos(2t-4u)}{-4}\right]_{u=0}^{t}$$

$$= \frac{1}{4}\left[\sin(2t)\left[t-0\right] + \frac{1}{4}\left[\cos(-2t) - \cos(2t)\right]\right]$$

$$= \frac{1}{4}\left[t\sin(2t) + \frac{1}{4}\left[\cos(2t) - \cos(2t)\right]\right]$$

$$\therefore\ L^{-1}\left[\frac{s}{s^4 + 8s^2 + 16}\right] = \frac{t}{4}\sin(2t)$$

Q.5(b) Find the Bilinear Transform which transform the points z = 2, i, -2 of z-plane into the points w = 1, i, -1 of the w-plane respectively. Also find fixed points of this transformation. **[6]**

(A) $z_1 = 2$, $z_2 = i$, $z_3 = -2$ and $w_1 = 1$ $w_2 = i$ and $w_3 = -1$

we know Bilinear Transform is

$$\frac{w - w_1}{w_1 - w_2} \cdot \frac{w_2 - w_3}{w_3 - w} = \frac{z - z_1}{z_1 - z_2} \cdot \frac{z_2 - z_3}{z_3 - z}$$

$$\therefore \quad \frac{w-1}{1-i} \cdot \frac{i+1}{-1-w} = \frac{z-2}{z-i} \cdot \frac{i+2}{-2-z}$$

$$\therefore \quad \frac{w-1}{1+w} \cdot \frac{1+i}{-1+i} = \frac{z-2}{z+2} \cdot \frac{2+i}{-2+i}$$

$$\therefore \quad \frac{w-1}{1+w} = \frac{z-2}{z+2} \cdot \frac{2+i}{-2+i} \times \frac{-1+i}{1+i} = \frac{(z-2)}{(z+2)} \cdot \frac{(4-3i)}{5}$$

$$\frac{w-1}{w+1} = \frac{4z - 8 - 3iz + 6i}{5z + 10}$$

By compondant dividendo

$$\frac{w - 1 + w + 1}{w - 1 - (w+1)} = \frac{4z - 8 - 3iz + 6i + 5z + 10}{4z - 8 - 3iz + 6i - 5z - 10}$$

$$\frac{2z}{-2} = \frac{9z + 2 - 3iz + 6i}{-z - 18 - 3iz + 6i} = \frac{(9-3i)z + (2+6i)}{-(1+zi)z - (18-6i)}$$

$$\therefore \quad w = \frac{(9+3i)z + (2+6i)}{(1+3i)z - (18-6i)} = \frac{3z(3-i) + (6i - 2i^2)}{z(1+3i) + (6i - 18)}$$

$$w = \frac{3z(3-i) + 2i(3-i)}{z(1+3i) + (6i + 18i^2)} = \frac{(3-i)(3z+2i)}{z(1+3i) + 6i(1+3i)}$$

$$w = \frac{(3-i)(3z+2i)}{(1+3i)(z+6i)} = \frac{(3-i)(3z+2i)}{(-i^2+3i)(z+6i)} = \frac{(3-i)(3z+2i)}{i(z+6i)(3-i)}$$

$$\therefore \quad w = \frac{3z+2i}{iz+6}$$

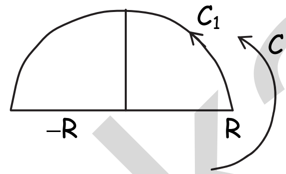
Q.5(c) Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{x^6+1}dx$ **[8]**

(A) Consider $\int_C \frac{z^2}{z^6+1}dz$ where C is large semi-circle as shown in figure

$$\therefore \quad \int_C \frac{z^2}{z^6+1}dz = \int_{-R}^{R} \frac{z^2}{z^6+1}dz + \int_{C_1} \frac{z^2}{z^6+1}dz \qquad \text{... (1)}$$

taking $\lim_{R\to\infty}$, along real axis y = 0

$\therefore \quad z = x + iy \Rightarrow z = x \qquad dz = dx$



and $\int_{C_1} \frac{z^2}{z^6+1}dz = 0$

$\therefore$ equation (1) gives,

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{x^6+1} = \int_C \frac{z^2}{z^6+1}dz \qquad \text{... (2)}$$

Here poles are $z^6 = -1 = e^{i\pi} = e^{i(\pi+2n\pi)} \Rightarrow z = e^{i(1+2n)\frac{\pi}{6}}$

n = 0, 1, 2, 3, 4, 5

$z_0 = e^{i\frac{\pi}{6}}$, $z_1 = e^{i\frac{3\pi}{6}}$, $z_2 = e^{i\frac{5\pi}{6}}$ are inside and $f(z) = \frac{z^2}{z^6+1}$

at $z = z_0$ Res. = $\left.\frac{z^2}{6z^5}\right|_{z=z_0} = \frac{1}{6}(z_0)^{-3}$

Similarly at $z = z_1$, Res = $\frac{1}{6}(z_1)^{-3}$, at $z = z_2$, Res = $\frac{1}{6}(z_2)^{-3}$

using C.R. Theorem for equation (2) we get

$$\int_{-\infty}^{\infty} \frac{x^2}{x^6+1}dx = 2\pi i\left\{\frac{1}{6}(z_0)^{-3} + \frac{1}{6}(z_1)^{-3} + \frac{1}{6}(z_2)^{-3}\right\}$$

$$= \frac{\pi i}{3}\left\{\left(e^{i\frac{\pi}{6}}\right)^{-3} + \left(e^{i\frac{3\pi}{6}}\right)^{-3} + \left(e^{i\frac{5\pi}{6}}\right)^{-3}\right\}$$

$$= \frac{\pi i}{3}\left\{\underset{0}{\underset{\downarrow}{\cos}}\frac{\pi}{2} - i\underset{1}{\underset{\downarrow}{\sin}}\frac{\pi}{2} + \underset{0}{\underset{\downarrow}{\cos}}\frac{3\pi}{2} - i\underset{-1}{\underset{\downarrow}{\sin}}\frac{3\pi}{2} + \underset{0}{\underset{\downarrow}{\cos}}\frac{5\pi}{2} - i\underset{1}{\underset{\downarrow}{\sin}}\frac{5\pi}{2}\right\}$$

$$= \frac{\pi i}{3}\{-i - i(-1) - i(1)\} = \frac{-\pi i^2}{3} = \frac{\pi}{3}$$

Q.6(a) Evaluate $\int_C \left(x^2 - 2ixy\right)dz$ **along** $y = 2x^2$ **From z = 0 to z = 3 + 18i** **[6]**

(A) $I = \int_C \left(x^2 - 2ixy\right)\left(dx + idy\right)$;

given $y = 2x^2$

dy = 4xdx

$$= \int_{0+0i}^{3+18i} \left[x^2 - 2ix\cdot 2x^2\right]\left[dx + i4x\,dx\right] \qquad = \int_0^3 \left[x^2 - 4ix^3 + 4ix^3 + 16x^4\right] dx$$

$$= \int_0^3 \left(x^2 + 16x^4\right)dx \quad = \left\{\frac{x^3}{3} + \frac{16x^5}{5}\right\}_0^3 \qquad = 9 + 16\left(\frac{243}{5}\right) - 0$$

$$\therefore \quad \int_C \left(x^2 - 2ixy\right)dz = \frac{3933}{5}$$

Q.6(b) Obtain the complex form of Fourier series for the function f(x) = e^{4x} in 0 < x < 4 **[6]**

(A) We know complex form of Fourier series in c to c + 2*l* is

$$f(x) = \sum_{n=-\infty}^{\infty} An\, e^{i\left(\frac{n\pi x}{l}\right)} \quad \text{where} \quad An = \frac{1}{2l}\int_{x=c}^{c+2l} f(x)e^{-i\left(\frac{n\pi x}{l}\right)}dx$$

On equating the interval (0, 4) with (c, c + 2*l*)

c = 0, c + 2*l* = 4, u = 4 ∴ ℓ = 2

∴ Complex form of Fourier series (0, 4) is

$$f(x) = \sum_{n=-\infty}^{\infty} An\; e^{i\left(\frac{n\pi x}{2}\right)} \qquad \dots (1)$$

$$\text{where} \quad An = \frac{1}{2\cdot 2}\int_{x=0}^{4} f(x)e^{-i\left(\frac{n\pi x}{2}\right)}dx \qquad = \frac{1}{4}\int_{x=0}^{4} e^{4x}\, e^{-i\left(\frac{n\pi x}{2}\right)}dx$$

$$\therefore \qquad An = \frac{1}{4}\int_{x=0}^{4} e^{(8-in\pi)\frac{x}{2}}\,dx \qquad = \frac{1}{4}\left.\frac{e^{(8-in\pi)\frac{x}{2}}}{(8-in\pi)\frac{1}{2}}\right|_{x=0}^{4}$$

$$= \frac{1}{2}\frac{8+in\pi}{(8-in\pi)(8+in\pi)}[e^{2(8-in\pi)} - e^0]$$

$$= \frac{1}{2}\frac{8+in\pi}{64+n^2\pi^2}\left[e^{16}e^{-2in\pi} - 1\right] \quad = \frac{8+in\pi}{2\left(64+n^2\pi^2\right)}\left(e^{16}-1\right) \quad [\because\; e^{\pm in\pi}\;(-1)^n]$$

$$= \frac{e^4\left(8+in\pi\right)}{64+n^2\pi^2}\left(\frac{e^4 - e^{-4}}{2}\right) \qquad = \frac{e^4\left(8+in\pi\right)}{64+4^2\pi^2}\sinh\left(4\right) \qquad \dots (2)$$

From (1) and (2)

$$f(x) \quad = \sum_{n=-\infty}^{\infty} \frac{e^4\,(8+in\pi)\sinh(4)}{64+n^2\pi^2}e^{i\left(\frac{n\pi x}{2}\right)}$$

$$\therefore\; f(x) = e^4\sinh\left(4\right)\sum_{n=-\infty}^{\infty}\left[\frac{8+in\pi}{64+n^2\pi^2}\right]e^{i\left[\frac{n\pi x}{2}\right]}$$

Q.6(c) Find half range sine series of the function **[8]**
$f(x) = x(3-x)$ in $0 \le x \le 3$
Hence prove that

(i) $\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \ldots = \frac{\pi^6}{960}$ **(ii)** $\sum_{n=1}^{\infty} \frac{1}{(n)^6} = \frac{\pi^6}{945}$

(A) We know half range sine series in (0, 3) is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{(n\pi x)}{3} \quad \ldots(1)$$

where $b_n = \frac{2}{3}\int_{x=0}^{3} f(x)\sin\left(\frac{n\pi x}{3}\right)dx = \frac{2}{3}\int_{x=0}^{3}\left(3x - x^2\right)\sin\left(\frac{n\pi x}{3}\right)dx$

$$b_n = \frac{2}{3}\left[\left(3x\text{-}x^2\right)\left[\frac{-\cos(n\pi x/3)}{n\pi/3}\right]\text{-}(3\text{-}2x)\left[\frac{\text{-}\sin(n\pi x/3)}{(n\pi/3)^2}\right]+(\text{-}2)\frac{\cos(n\pi x/3)}{(n\pi/3)}\right]_{x=0}^{3}$$

$$b_n = \frac{2}{3}\left[\frac{\text{-}3}{n\pi}\left(3x\text{-}x^2\right)\cos\left(\frac{n\pi x}{3}\right)+\frac{9}{n^2\pi^2}(3\text{-}2x)\sin\left(\frac{n\pi x}{3}\right)\text{-}2\times\frac{27}{n^3\pi^3}\cos\left(\frac{n\pi x}{3}\right)\right]_{x=0}^{3}$$

$$b_n = \frac{2}{3}\left[\frac{-3}{n\pi}[0-0]+\frac{9}{n^2\pi^2}[0-0]-\frac{54}{n^3\pi^3}\left[(-1)^n-1\right]\right]$$

$$b_n = \frac{36(1-(-1)^n)}{n^3\pi^3} \quad \ldots(2)$$

From (1) and (2)

$$F(x) = \sum_{n=1}^{\infty}\frac{36(1-(-1)^n)}{n^3\pi^3}\cdot\sin\left(\frac{n\pi x}{3}\right) = \frac{36}{\pi^3}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^3}\right]\sin\left(\frac{n\pi x}{3}\right)$$

Now Parse Val's identity of (1) is

$$\sum_{n=1}^{\infty}\left(b_n\right)^2 = \frac{2}{3}\int_{x=0}^{3}\left[f(x)\right]^2dx = \frac{2}{3}\int_{x=0}^{3}x^2\left(9-\sigma x+x^2\right)dx$$

$$\sum_{n=1}^{\infty}\frac{36\times36}{n^6\pi^6}\left[1+1-2(-1)^n\right] = \frac{2}{3}\int_{x=0}^{3}9x^2-\sigma x^3+x^4\,dx$$

$$\frac{36\times36}{\pi^6}\sum_{n\pi}^{\infty}\frac{1-(-1)^n}{n^6} = \frac{1}{3}\left[\frac{9x^3}{3}-\frac{\sigma x^4}{4}+\frac{x^5}{5}\right]_{x=0}^{3}$$

$$\frac{36\times36}{\pi^6}\left[\frac{2}{1^6}+a\ell\frac{2}{3^6}+a\ell\frac{2}{5^6}\ldots\right] = \frac{1}{3}\left[\frac{95}{3}-\frac{1}{2}3^4+\frac{1}{5}\cdot35\right]$$

$$\frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}\ldots = \frac{\pi^6}{36\times36\times2}\times\frac{9\times9}{3}\left[1-\frac{3}{2}+\frac{3}{5}\right]$$

$$= \frac{\pi^6}{2\times48}\times\frac{10-15+16}{10} = \frac{\pi^6\cdot1}{960}$$

$$\therefore \quad \frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}\ldots = \frac{\pi^6}{960} \qquad \ldots(3)$$

$$\text{Let } s = \sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\frac{1}{6^6}+\ldots \qquad \ldots(4)$$

$$s = \left(\frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}+\ldots\right)+\left(\frac{1}{(2\times1)^6}+\frac{1}{(2\times2)^6}+\frac{1}{(2\times3)^6}+\ldots\right)$$

$$s = \frac{\pi^6}{960}+\frac{1}{64}\left[\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}\ldots\right]$$

$$s = \frac{\pi^6}{960}+\frac{1}{64}\times 5$$

$$s-\frac{s}{64} = \frac{\pi^6}{960}$$

$$\frac{63}{64}s = \frac{\pi^6}{960}$$

$$\therefore \quad s = \frac{\pi^6}{960}\times\frac{64}{63} = \frac{\pi^6}{15\times63} = \frac{\pi^6}{945}$$

$$\therefore \quad \sum_{n\pi}^{\infty}\left(\frac{1}{n^6}\right) = \frac{\pi^6}{945}$$

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