S.E. Sem. III [MECH/AUTO]

Applied Mathematics - III

Time : 3 Hrs.] Prelim Paper Solution **[Marks : 80**

Q.1(a) Find Laplace transform of te^{3t} cost. **[5]**

Ans.: $L\{\cos t\} = \dfrac{s}{s^2+1}$, $L\{t\cos t\} = (-1)\dfrac{d}{ds}\dfrac{s}{s^2+1}$

$\therefore$ $L\{t\cos t\} = -\left\{\dfrac{s^2+1-s(2s)}{(s^2+1)^2}\right\} = \dfrac{s^2-1}{(s^2+1)^2}$

by F.S.T.

$$L\left\{t\,e^{3t}\cos t\right\} = \frac{(s-3)^2-1}{\left[(s-3)^2+1\right]^2}$$

Q.1(b) Evaluate $\oint_c \dfrac{z-1}{z^2+2z+5}dz$, where c is |z + 1 + i| = 2. **[5]**

Ans.: Let $I = \int_c \dfrac{z-1}{z^2+2z+5}dz$ $\qquad c : |z+1+i| = 2$

for poles, $z^2 + 2z + 5 = 0$

$\Rightarrow$ $z = -1+2i$, $\quad z = -1-2i$

$|-1+2i+1+i| = |0+3i| = 3 > 2$ outside

$|-1-2i+1+i| = |0-i| = 1 < 2$ $\Rightarrow$ pole $z = -1-2i$ is inside

at $z = -1-2i$ Residue $= \left.\dfrac{z-1}{2z+2}\right|_{z=-1-2i} = \dfrac{-1-2i-1}{-2-4i+2}$

$= \dfrac{-2-2i}{-4i} = \dfrac{1+i}{2i}$

by Residue Theorem $\int_c \dfrac{z-1}{z^2+2z+5} = 2\pi i\left(\dfrac{1+i}{2i}\right) = \pi(1+i)$

Q.1(c) Show that f(z) = sinh z is analytic. Hence find its derivative. **[5]**

Ans.: $f(z) = \sinh z = \sinh(x+iy) = \sinh x \cosh iy + \cosh x \sinh iy$

$u + iv = \sinh x \cos y + i\cosh x \sin y$

$\Rightarrow$ $u = \sinh x \cos y$

$u_x = \cosh x \cos y$ …(1)

$u_y = \sinh x \sin y$ …(2)

$v = \cosh x \sin y$

$v_x = \sinh x \sin y$ …(3)

$v_y = \cosh x \cos y$ …(4)

From (1), (4) and (2), (3) $u_x = v_y$ $v_x = -u_y$

$\therefore$ f(z) = sinh z is analytic.

Now $f'(z) = u_x + iv_x\big|_{x=z,\,y=0} = \cosh x\cos y + i\sinh x\sin y\big|_{x=z,\,y=0}$

$f'(z) = \cosh z$

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Q.1(d) Compute spearman's rank correlation for the data : **[5]**

X :	**18**	**20**	**34**	**52**	**12**
Y :	**39**	**23**	**35**	**18**	**46**

Ans.:

X	Y	R_x	R_y	$d = R_x - R_y$	d^2
18	39	4	2	2	4
20	23	3	4	−1	1
34	35	2	3	−1	1
52	18	1	5	−4	16
12	46	5	1	4	16

$\sum d^2 = 38$

$$R = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6(38)}{5(24)} \Rightarrow R = -0.9$$

Q.2(a) Show that the function $\omega = \frac{4}{z}$ transforms the straight line x = c in the z-plane into circle is w-plane. Find its centre and radius. **[6]**

Ans.: $x = c$ ...(1)

Given $\omega = \frac{4}{z} \Rightarrow z = \frac{4}{\omega} \Rightarrow x + iy = \frac{4}{u+iv}$

$\therefore \quad x + iy = \frac{4u}{u^2+v^2} - \frac{i4v}{u^2+v^2} \Rightarrow x = \frac{4u}{u^2+v^2}$

using this in (1) $\frac{4u}{u^2+v^2} = c \Rightarrow u^2 + v^2 - \frac{4u}{c} = 0$

centre $= \left(\frac{z}{c}, 0\right)$, rad $= \frac{2}{c}$

Q.2(b) Show that $\int_0^\infty e^{-t}\int_0^t \frac{\sin u}{u}\,du\,dt = \frac{\pi}{4}$ **[6]**

Ans.: $\int_0^\infty e^{-t}\int_0^t \frac{\sin u}{u}\,du\,dt = L\left\{\int_0^t \frac{\sin u}{u}\,du\right\}\Bigg|_{s=1}$

Now

$$L\left\{\int_0^t \frac{\sin u}{u}\,du\right\} = \frac{1}{s}L\left\{\frac{\sin u}{u}\right\} = \frac{1}{s}\int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= \frac{1}{s}\left\{\tan^{-1} s\right\}_{s=s}^{s=\infty} = \frac{1}{s}\left[\frac{\pi}{2} - \tan^{-1} s\right]$$

$\therefore$ at s = 1,

$$= \frac{1}{1}\left[\frac{\pi}{2} - \tan^{-1} 1\right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

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Q.2(c) **Obtain fourier series for** $f(x) = \begin{cases} 1+\dfrac{2x}{\pi} & -\pi \le x \le 0 \\ 1-\dfrac{2x}{\pi} & 0 \le x \le \pi \end{cases}$ **[8]**

Hence deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+....$

Ans.: f(x) is even function. $\Rightarrow$ bn = 0

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) \qquad ...(1)$$

$$a_0 = \frac{1}{2\pi}2\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)dx = \frac{1}{\pi}\left[x-\frac{x^2}{\pi}\right]_0^{\pi} \qquad \Rightarrow a_0 = 0$$

$$a_n = \frac{1}{\pi}2\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos(nx)\,dx = \frac{2}{\pi}\left\{\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin(nx)}{n}\right)-\left(-\frac{2}{\pi}\right)\left(\frac{-\cos(nx)}{n^2}\right)\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\left[0-\frac{2}{\pi n^2}(-1)^n\right]-\left[0-\frac{2}{\pi n^2}\right]\right\}$$

$$a_n = \frac{4}{n^2\pi^2}\left[1-(-1)^n\right]$$

using this in (1)

$$f(x) = \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left[1-(-1)^n\right]\cos(nx)$$

for deduction put x = 0 and note that f(0) = 1

$$1 = \sum_{n=1}^{\infty}\frac{4}{\pi^2 n^2}\left[1-(-1)^n\right]$$

$$1 = \frac{4}{\pi^2}\left[\frac{2}{1^2}+0+\frac{2}{3^2}+0+\frac{2}{5^2}+...\right] \qquad \Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...$$

Q.3(a) **Evaluate** $\oint_c \dfrac{e^{kz}}{z}dz$ **, where c : |z| = 1.** **[6]**

Hence deduce that $\int_0^{\pi} e^{k\sin\theta}\cos(k\sin\theta)\,d\theta = \pi$

Ans.: $I = \int_c \dfrac{e^{kz}}{z}dz$, c : |z| = 1

pole z = 0 is inside |z| = 1

$\therefore \int_c \dfrac{e^{kz}}{z}dz = 2\pi i\, f(0)$, where $f(z) = e^{kz}$

$= 2\pi i$ (1)

put $z = e^{i\theta}$ $\Rightarrow$ $dz = ie^{i\theta}d\theta$, limits 0 to 2π

$$\therefore (1) \Rightarrow \int_0^{2\pi}\frac{e^{ke^{i\theta}}}{e^{i\theta}}\,i\,e^{i\theta}d\theta = 2\pi i$$

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$$i\int_0^{2\pi} e^{k(\cos\theta + i\sin\theta)} d\theta \quad = 2\pi i$$

$$\int_0^{2\pi} e^{k\cos\theta}\left[\cos(k\sin\theta) + i\sin(k\sin\theta)\right] d\theta \ = 2\pi$$

$$\int_0^{2\pi} e^{k\cos\theta}\cos(1 < \sin\theta)\, d\theta + \int_0^{\pi} e^{k\cos\theta}\sin(k\sin\theta) \ = 2\pi + 0i$$

$$\Rightarrow \int_0^{2\pi} e^{k\cos\theta}\cos(k\sin\theta)\, d\theta \quad = 2\pi$$

$$2\int_0^{2\pi} e^{k\cos\theta}\cos(k\sin\theta)\, d\theta \quad = 2\pi$$

$$\Rightarrow \int_0^{2\pi} e^{k\cos\theta}\cos(k\sin\theta)\, d\theta \quad = \pi$$

Q.3(b) **For the lines of regression 6y – 5x = 90, 15 x – 8y = 130 and σ_x^2 = 16** **[6]**

Find (i) $\overline{x}, \overline{y}$ (ii) r (iii) σ_y

Ans.: given 6y – 5x = 90, 15x – 8y = 130

solving we get x = 30, y = 40

i.e. $\overline{x}$ = 30, $\overline{y}$ = 40

$$y = \frac{5}{6}x + 15 \qquad \Rightarrow b_{yx} = \frac{5}{6}$$

$$x = \frac{8}{15}y + \frac{130}{15} \qquad \Rightarrow b_{xy} = \frac{8}{15}$$

$$\therefore \quad r = \pm\sqrt{b_{xy}\, b_{yx}} = +\sqrt{\frac{5}{6}\cdot\frac{8}{15}} \quad \Rightarrow r = 0.67$$

given $\sigma_x^2 = 16 \quad \Rightarrow \sigma_x = 4$

$$b_{xy} = \frac{8}{15} \quad \Rightarrow r\frac{\sigma_x}{\sigma_y} = \frac{8}{15} \quad \Rightarrow \frac{2}{3}\left(\frac{4}{\sigma_y}\right) = \frac{8}{15}$$

$6_y = 5$

Q.3(c) **Solve the differential equation $\dfrac{dy}{dx} + 2y + \displaystyle\int_0^t y\,dt$ = sin t using Laplace transform give y(0) = 1** **[8]**

Ans.:

$$L\{y'(t)\} + 2L\{y(t)\} + L\left\{\int_0^t y(t)\,dt\right\} \quad = L\{\sin t\}$$

$$5y(s) - 1 + 2y(s) + \frac{1}{s}y(s) \quad = \frac{1}{s^2+1}$$

$$\left(s + 2 + \frac{1}{s}\right)y(s) \quad = \frac{1}{s^2+1} + 1 \quad = \frac{s^2+2}{s^2+1}$$

$$\frac{(s^2+2s+1)}{s}y(s) = \frac{s^2+2}{s^2+1} \Rightarrow y(s) = \frac{s(s^2+2)}{(s+1)^2(s^2+1)}$$

$$y(s) = \frac{a}{s+1}+\frac{b}{(s+1)^2}+\frac{cs+d}{s^2+1}, \ a = 1, b = -\frac{3}{2}, c = 0, d = \frac{1}{2}$$

$$\therefore \quad y(s) = \frac{1}{s+1}-\frac{3}{2}\frac{1}{(s+1)^2}+\frac{1}{2}\frac{1}{s^2+1}$$

taking inverse Laplace,

$$y(t) = e^{-t}-\frac{3}{2}te^{-t}+\frac{1}{2}\sin t$$

Q.4(a) **Find Laurentz's series for f(z) = $\frac{2}{(z-1)(z-2)}$ indicating region of convergence.** **[6]**

Ans.: $$f(z) = \frac{2}{(z-1)(z-2)} = \frac{a}{z-1}+\frac{b}{z-2} = -\frac{2}{z-1}+\frac{2}{z-2} \quad \text{...(1)}$$

for Laurent's series ROC is (1) |z| > 1, |z| > 2 (ii) 1 < |z| < 2

case (1) : |z| > 1, |z| > 2

$$(1) \Rightarrow f(z) = \frac{-2}{z\left(1-\frac{1}{z}\right)}+\frac{2}{z\left(1-\frac{2}{z}\right)} = -\frac{2}{z}\left(1-\frac{1}{z}\right)^{-1}+\frac{2}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$f(z) = \frac{-2}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\ldots\right]+\frac{2}{z}\left[1+\frac{2}{z}+\frac{4}{z^2}+\ldots\right]$$

case (2) : 1 < |z| < 2

$$(1) \Rightarrow f(z) = \frac{-2}{z\left(1-\frac{1}{z}\right)}-\frac{2}{2\left(1-\frac{z}{2}\right)} = -\frac{2}{z}\left(1-\frac{1}{z}\right)^{-1}-\left(1-\frac{z}{2}\right)^{-1}$$

$$f(z) = \frac{-2}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\ldots\right]-\left[1+\frac{z}{2}+\frac{z^2}{4}+\ldots\right]$$

Q.4(b) **Show that cos x = $8\pi\sum_{m=1}^{\infty}\frac{m}{4m^2-1}\sin(2mx)$, If $0 < x < \pi$** **[6]**

Ans.: Half range sine series $f(x) = \sum_{n=1}^{\infty} bn\sin(nx)$ ...(1)

$$b_n = \frac{2}{\pi}\int_0^{\pi}\cos x\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi}\left[\frac{\sin(nx+x)+\sin(nx-x)}{2}\right]dx$$

$$= \frac{1}{\pi}\left\{\frac{-\cos(nx+x)}{n+1}-\frac{\cos(nx-x)}{n-1}\right\}_0^{\pi} = \frac{1}{\pi}\left\{\left[\frac{(-1)^n}{n+1}+\frac{(-1)^n}{n-1}\right]-\left[-\frac{1}{n+1}-\frac{1}{n-1}\right]\right\}$$

$$b_n = \frac{1}{\pi}\left[1+(-1)^n\right]\left[\frac{1}{n+1}+\frac{1}{n-1}\right] = \frac{2n}{\pi(n^2-1)}\left[1+(-1)^n\right]$$

$n \neq 1$

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Now $b_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x \, dx \quad = \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx$

$= \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi} \quad = -\frac{1}{2\pi}[1-1] \quad \Rightarrow \quad b_1 = 0$

$\Rightarrow \quad \cos x = \sum_{n=2}^{\infty} \frac{2n}{\pi(n^2-1)}\left[1+(-1)^n\right]\sin(nx)$

put n = 2m

$\therefore \quad \cos x = \sum_{2m=2}^{\infty} \frac{2(2m)}{\pi(4m^2-1)}\left[1+(-1)^{2m}\right]\sin(2mx)$

$\therefore \quad \cos x = \frac{8}{\pi}\sum_{m=1}^{\infty} \frac{m}{4m^2-1}\sin(2mx)$

Q.4(c) Find Bilinear transformation which maps the points 1, i, –1 onto the points i, 0, –i. Hence find fixed points and image of |z| < 1. **[8]**

Ans.: Consider B. I. $\omega = \frac{az+b}{cz+d}$ ...(1)

given $z = 1, \omega = i$ (1)

$\Rightarrow \quad i = \frac{a+b}{c+d} \quad \Rightarrow \quad ic + id = a + b$ ...(2)

given $z = i, \omega = 0$ (1)

$\Rightarrow \quad 0 = \frac{ia+b}{ic+d} \quad \Rightarrow b = -ia$ ...(3)

given $z = -1, \omega = -i$ (1)

$\Rightarrow \quad -i = \frac{-a+b}{-c+d} \quad \Rightarrow \quad ic - id = -a + b$ ...(4)

(2) + (4) $\Rightarrow 2ic = 2b \quad \Rightarrow \quad 2ic = -2\,ia \quad \Rightarrow \quad c = -a$ ...(5)

(2) – (4) $\Rightarrow 2id = 2a \quad \Rightarrow \quad d = -ia$ ...(6)

using (3), (5), (6), in (1)

$\omega = \frac{az-ia}{-az-ia} \quad \Rightarrow \quad \omega = \frac{i-z}{i+z}$ ...(7)

For fixed points w = z, (7) $\Rightarrow \quad z = \frac{i-z}{i+z}$

$z^2 + iz = i - z \quad \Rightarrow \quad z^2 + (i+1)z - i = 0$

$z = \frac{-i-1 \pm \sqrt{-1+1+2i+4i}}{2} \quad = \frac{-i-1\pm\sqrt{6i}}{2}$

to find image of |z| < 1 : (7) $\Rightarrow \quad i\omega + \omega z = i - z \Rightarrow (1+\omega)z = i - i\omega$

$z = \frac{i-i(u+iv)}{1+u+iv}$

$\therefore \quad |z| < 1 \quad \Rightarrow \quad \left|\frac{i-iu+v}{1+u+iv}\right| < 1$

$|i - iu + v| = |1 + u + iv|$

$$\sqrt{v^2+(1-u)^2} < \sqrt{(1+u)^2+v^2}$$

$\Rightarrow$ $v^2 + 1 - 2u + u^2 < 1 + 2u + u^2 + v^2 \Rightarrow 0 < 4u$

$\Rightarrow$ $0 < u$

Q.5(a) Solve using Bender-Schmidt method $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t}$ = 0, subject to the conditions [6]

$u(0, t) = 0$, $u(1, t) = 0$, $u(x, 0) = \sin \pi x$, $0 \le x \le 1$

Ans.: a = 1, take h = 0.2 $\Rightarrow$ x : 0, 0.2 0.4 0.6 0.8 1

$$k = \frac{a}{2}h^2 = 0.02$$

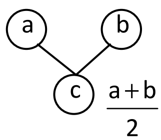
t : 0, 0.02 0.04 0.06 0.08 0.10

$x \to h = 0.2$

t $\downarrow$ k = 0.02

t \ x	0.0	0.2	0.4	0.6	0.8	1
0.00	0	0.5878	0.9511	0.9511	0.5878	0
0.02	0					0
0.04	0					0
0.06	0					0
0.08	0					0
0.10	0					0

using Bender-Schmidt formula $c = \frac{(a+b)}{2}$



t \ x	0	0.2	0.4	0.6	0.8	1
0	0	0.5878	0.9511	0.9511	0.5878	0
0.02	0	0.4756	0.7695	0.7695	0.4756	0
0.04	0	0.3848	0.6225	0.6225	0.3848	0
0.06	0	0.3113	0.5036	0.5036	0.3113	0
0.08	0	0.2518	0.4074	0.4074	0.2518	0
0.10	0	0.2037	0.3296	0.3296	0.2037	0

Q.5(b) Determine the solution of one-dimensional heat equation under the boundary conditions, $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ [6]

$u(0,t) = 0$, $u(\ell,t) = 0$, $u(x,0) = x$, $(0<x<\ell)$, ℓ being length of the rod.

Ans.: Solution of heat flow is given by

$$u = \left(c_1 \cos mx + c_2 \sin mx\right) e^{-m^2c^2t} \qquad \text{...(1)}$$

given u = 0 when x = 0

(1) $\Rightarrow$ $0 = c_1 e^{-m^2c^2t}$ $\Rightarrow$ $c_1 = 0$

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$\therefore$ (1) $\Rightarrow$ $u = c_2 \sin mx\ e^{-m^2c^2t}$ ...(2)

given when $x = \ell$, $u = 0$

(2) $\Rightarrow 0 = c \sin m\ell\ \ e^{-m^2c^2t}$ $\Rightarrow$ $\sin m\ell = 0$ $\Rightarrow m\ell = n\pi$

$\Rightarrow$ $m = \dfrac{n\pi}{\ell}$

(2) $\Rightarrow$ $u = c_2 \sin\left(\dfrac{n\pi x}{\ell}\right)e^{-\frac{n^2\pi^2}{\ell^2}c^2t}$ ...(3)

adding all above solutions for n = 1, 2, … we get the general solution

$$u = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)e^{-\frac{n^2\pi^2c^2t}{\ell^2}} \quad ...(3)$$

given when t = 0, u = x

$$(3) \Rightarrow x = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)$$

which is half range series in $(0, \ell)$

$$b_n = \frac{2}{\ell}\int_0^{\ell} x \sin\left(\frac{n\pi x}{\ell}\right)dx = \frac{2}{\ell}\left\{(x)\left[\frac{-\cos\left(\frac{n\pi x}{\ell}\right)}{\frac{n\pi}{\ell}}\right] - (1)\left[\frac{-\sin\left(\frac{n\pi x}{\ell}\right)}{\frac{n^2\pi^2}{\ell^2}}\right]\right\}_0^{\ell}$$

$$= \frac{2}{\ell}\left\{\left[-\frac{\ell}{\frac{n\pi}{\ell}}(-1)^n - 0\right] - [0 - 0]\right\} = -\frac{2\ell}{n\pi}(-1)^n$$

using this in (3) we get,

$$u = \sum_{n=1}^{\infty} \frac{-2\ell}{n\pi}(-1)^n \sin\left(\frac{nx\pi}{\ell}\right)e^{\frac{-n^2\pi^2c^2t}{\ell^2}}$$

Q.5(c) Find inverse Laplace Transform of **[8]**

(i) $\log\left(1 + \dfrac{a^2}{s^2}\right)$ **(ii)** $\dfrac{e^{-s}}{s^2 + s + 1}$

Ans.: (i) Let $f(s) = \log\left(1 + \dfrac{a^2}{s^2}\right) = \log\left(\dfrac{s^2 + a^2}{s^2}\right)$

$f(s) = \log(s^2 + a^2) - \log s^2$

$\Rightarrow$ $f'(s) = \dfrac{2s}{s^2 + a^2} - \dfrac{2}{s}$

taking L^{-1} we get

$L^{-1}\{f'(s)\} = 2\cos at - 2$

$\Rightarrow$ $-t\, f(t) = 2\cos at - 2$

$\Rightarrow$ $f(t) = \dfrac{2(1 - \cos at)}{t}$

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(ii) Let $f(s) = \dfrac{1}{s^2+s+1} = \dfrac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$

$\therefore \quad L^{-1}\{F(s)\} = L^{-1}\left\{\dfrac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\}$

$\Rightarrow \quad F(t) = e^{-\frac{1}{2}t} L^{-1}\left\{\dfrac{1}{s^2+\frac{3}{4}}\right\} = \dfrac{e^{-\frac{1}{2}t}}{\frac{\sqrt{3}}{2}} \sin\left(\dfrac{\sqrt{3}}{2}t\right)$

we know that,

$L^{-1}\{e^{-as} f(s)\} = f(t-a)\, H(t-a)$

$\therefore \quad L^{-1}\left\{\dfrac{e^{-s}}{s^2+s+1}\right\} = \dfrac{2}{\sqrt{3}} e^{-\frac{1}{2}(t-1)} \sin\left[\dfrac{\sqrt{3}}{2}(t-1)\right]\cdot H(t-1)$

Q.6(a) Obtain complex form of fourier series for $F(x) = e^{\alpha x}$, in $(-\pi, \pi)$ where a is not an integer. **[6]**

Ans.: Complex form of fourier series is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad \ldots(1)$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax} e^{-inx}\, dx = \frac{1}{2\pi}\left\{\frac{e^{ax-inx}}{a-in}\right\}_{-\pi}^{\pi} = \frac{1}{2\pi}\frac{(a+in)}{(a+in)(a-in)}\left\{e^{a\pi}e^{-in\pi} - e^{-a\pi}e^{in\pi}\right\}$$

$$= \frac{1}{2\pi}\frac{(a+in)}{(a^2+n^2)}(-1)^n\left(e^{a\pi}-e^{-a\pi}\right) = \frac{(a+in)}{\pi(a^2+n^2)}(-1)^n \sinh(a\pi)$$

using this in (1)

$$e^{ax} = \sum_{n=-\infty}^{\infty}\frac{(a+in)}{\pi(a^2+n^2)}(-1)^n \sinh(a\pi)\, e^{inx}$$

Q.6(b) Fit a curve $y = a \cdot b^x$ to the following data, using method of least squares. **[6]**

X :	2	3	4	5	6
Y :	144	172.8	207.4	248.8	298.5

Ans.: $y = a \cdot b^x$ (taking log) …(1)

$\log y = \log a + x \log b$

$\Rightarrow \quad \sum \ln g = n \ln a + \ln b \sum x \qquad \ldots(2)$

$\sum x \ln y = \ln a \sum x + \ln b \sum x^2 \qquad \ldots(3)$

(2) $\Rightarrow$ $26.58 = 5A + 20B$

(3) $\Rightarrow$ $108.51 = 20A + 90B$

where A = ln a, B = ln b

Solving: ln a = 4.62

B = 0.179 a = 101.49

$\therefore$ (1) $\Rightarrow$ ln b = 0.179

$y = (101.49)(1.196)^x$ b = 1.196

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Q.6(c) **(i)** **Evaluate** $\int_0^{2\pi} \frac{d\theta}{5+3\sin\theta}$ **using Residue Theorem.** **[4]**

(ii) **Evaluate** $\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$ **using Residue Theorem.** **[4]**

Ans.: (i) $I = \int_0^{2\pi} \frac{d\theta}{5+3\sin\theta}$ = put $z = e^{i\theta} \Rightarrow d\theta = \frac{dz}{iz}$

$\sin\theta = \frac{z^2-1}{2iz}$

$\therefore \quad I = \int_c \frac{\frac{dz}{iz}}{5+3\left(\frac{z^2-1}{2i^z}\right)} = \int_c \frac{2}{3z^2+10iz-3} dz$

poles are $z = -\frac{i}{3}$, $z = -3i$

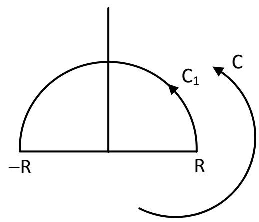
$z = \frac{-i}{3}$ lies inside $|z| = 1$

at $z = \frac{-i}{3}$, $\quad \text{Res} = \left.\frac{2}{6z+10i}\right|_{z=-\frac{i}{3}} = \frac{2}{-2i+10i} = \frac{1}{4i} = \frac{1}{4i}$

by Residue Theorem,

$I = 2\pi i\left(\frac{1}{4i}\right) = \frac{\pi}{2}$

(ii) Consider $\int_c \frac{dz}{z^2+1}$ where c is large semicircle as shown in figure.



$\therefore \quad \int_{-R}^{R} \frac{dz}{z^2+1} + \int_{C_1} \frac{dz}{z^2+1} = \int_c \frac{dz}{z^2+1} \quad$ ...(1)

taking limit as $R \to \infty$

along real axis $y = 0 \Rightarrow z = x$ and $\int_{c_1} \frac{dz}{z^2+1} = 0$

$\therefore \quad (1) \Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \int_C \frac{dz}{z^2+1}$,

poles are $z = \pm i$ but $z = i$ lies inside

at $z = i \quad \text{Res} = \left.\frac{1}{2z}\right|_{z=i} = \frac{1}{2i}$

$\therefore$ by residue theorem

$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = 2\pi i\left(\frac{1}{2i}\right) = \pi$

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Vidyalankar