S.E. Sem. III [CMPN]

Discrete Mathematics

Time : 3 Hrs.] Prelim Paper Solution [Marks : 80

Q.1(a) Define the following : [5]

(a) Disjoint set **(b) Partial or division relation**
(c) Symmetric difference **(d) Antisymmetric relation**
(e) Cartesian product

Ans.: (a) Disjoint set : Two or more set which have no elements in common are known as disjoint sets. For example, A = (a, b, c) and B = (d, e, f) are disjoints sets.

(b) Partial order relation : A relation R on set A is partial ordered if it is reflexive, an antisymmetric and transitive.

(c) Symmetric difference : The symmetric difference of two sets is the set of elements which are in either of the sets and not in their intersection. The symmetric difference of the sets A and B is commonly denoted by

$A \Delta B$ or $A \ominus B$ or $A + B$

For example, the symmetric difference of the sets {1, 2, 3} and {3, 4} is {1, 2, 4}

(d) Antisymmetric relation : If xRy and yRx $\Rightarrow x = y$

(e) Cartesian product : Cartesian product is a mathematical operation which returns a set (or product set or simply product) from multiple sets. That is, for sets A & B, the Cartesian product $A \times B$ is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. Products can be specified using set builder notation.
e.g. $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$

Q.1(b) Let A = {1, 2, 3, 4, 6} and let R be the relation on A defined by "x divides y" written x/y. (Note x/y if there exists an integer z such that xz = y) [5]

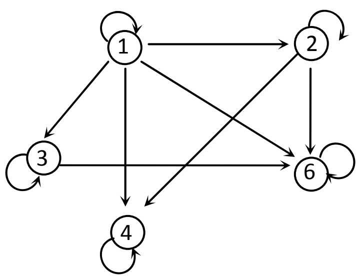
(i) Write R as a set of ordered pairs.
(ii) Draw its directed graph.
(iii) Find the inverse relation of R.

Ans.: A = {1, 2, 3, 4, 6}
R = "x divides y" (x/y) such that xz – y i.e. x/y – 1/z

(i) R = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (2,4), (2, 6), (3, 6), (2, 2), (3, 3), (4, 4), (6, 6)}

(ii) R^{-1} = {(1, 1), (2, 1), (3, 1), (4, 1), (6, 1), (4, 2), (6, 2), (6, 3), (2, 2), (3, 3), (4, 4), (6, 6)}

(iii)



Q.1(c) Let (A, *) be an algebraic system, where * is a binary operation such that $\forall$a, b $\in$ A, a * b = a. **[5]**

(i) Show that * is associative. **(ii) Is * commutative?**

Ans.: (i) $\forall$ a, b, c $\in$ A

By definition,

$a * (b * c) = a * b = a$ … (1)

$(a * b) * c = a * c = a$ … (2)

From (1) and (2),

$a * (b * c) = (a * b) * c$

$\therefore$ * is associative.

(ii) If A has more than one element then

$a * b = a$ & $b * a = b$ (by definition)

$\therefore$ $a * b \neq b * a$

$\therefore$ * is not commutative.

Q.1(d) Let m be the positive integer greater than 1. **[5]**
Show that the relation R = {(ab) | a =b(mod m)} i.e. aRb if and only if m divides a – b is equivalence relation on the set of integers.

Ans.: (a) Reflexive : We know a =b (mod m) if and only if m divides a – b.
Now a = 0 is divisible by m.
So $a \equiv a \pmod m$
$\therefore$ Congruence module m is reflexive.
Hence given relation R is reflexive.

(b) Symmetric : Suppose a = b (mod m) then (a–b) is divisible by m. So that a–b = km, whenever k is an integer.
If follows that $(b - a) = (-km)$
So that $b = a \pmod m$
So congruence module m is symmetric. Hence given relation R is symmetric.

(c) Transitive: Suppose $a \equiv b \pmod m$ and $b \equiv c \pmod m$ then m divides both a – b and b – c. Therefore there are integers k and l with $a - b = km$ and $b - c = lm$

$(a - b) + (b - c) = (k + l)m$

$a - c = (k + l)m$

$\therefore$ $a = c \pmod m$

$\therefore$ Congruence modulo m is transitive.
Hence given relation R is transitive.
$\therefore$ give relation R is Equivalence relation.

Q.2(a) Prove that $((A \cup B) \cap \overline{A}) \cup (\overline{B \cap A}) = (\overline{A \cap B})$. **[4]**

Ans.: L.H.S. $= ((A \cup B) \cap \overline{A}) \cup (\overline{B \cap A})$

$= [(A \cap \overline{A}) \cup (B \cap \overline{A})] \cup (\overline{B \cap A})$ distributive law

$= [\phi \cup B \cap \overline{A}] \cup (\overline{B} \cup \overline{A})$ $\overline{B \cap A} = \overline{B} \cup \overline{A}$ De Morgan's law & $A \cap \overline{A} = \phi$

$= (B \cap \overline{A}) \cup (\overline{B} \cup \overline{A})$

$= [B \cup (\overline{B} \cup \overline{A})] \cap [\overline{A} \cup (\overline{B} \cup \overline{A})]$ distributive law

$= [(B \cup \overline{B}) \cup \overline{A}] \cap [\overline{A} \cup \overline{B}]$ associative law

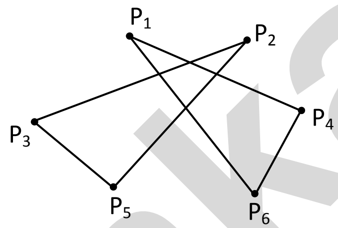
$= [U \cup \overline{A}] \cap [\overline{B} \cup \overline{A}]$ $B \cup \overline{B} = U$

$= U \cap [\overline{A} \cup \overline{B}]$ $\because$ U is universal set

$= \overline{A} \cup \overline{B}$

$= \overline{A \cap B}$ By De Morgan's law

Q.2(b) Describe formally the graph shown in figure. **[4]**



Ans.: Graph G = (V, E)

$\because$ There are six vertices.

$\therefore$ V = {P_1, P_2, P_3, P_4, P_5, P_6}

$\because$ There are six edges

$\therefore$ e = [{P_1, P_4}, {P_1, P_6}, {P_2, P_3}, {P_2, P_5}, {P_3, P_5},{P_4, P_6}]

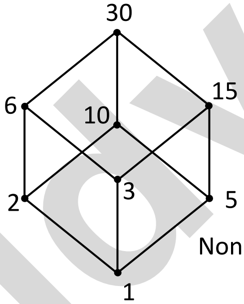
Q.2(c) Consider the partial order of divisibility on set A. **[6]**

Draw Hass diagram of the poset and determine which poset are linearly ordered.

(i) A = {1, 2, 3, 5, 6, 10, 15, 30} **(ii) A = {2, 4, 8, 16, 32}**

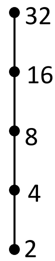
(iii) A = {3, 6, 12, 36, 72 **(iv) A = {1, 2, 3, 4, 5, 6, 10, 12, 15, 30, 60}**

Ans.: (i)



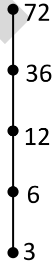
Nonlinearly Ordered

(ii)



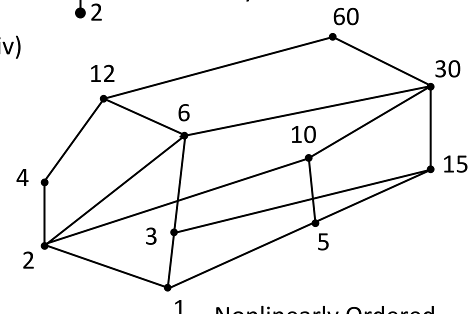
Linearly Ordered

(iii)



Linearly Ordered

(iv)



Nonlinearly Ordered

Note : Determining whether R is partial order from M_R

Q.2(d) **Let** $H = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **[6]**

be a parity check matrix. Decode the following word related to maximum likelihood technique (Decoding function) associated with e_H. Decode the following : (i) 10100 (ii) 01101 (iii) 11011

Ans.: Here, m = 2, n = 5

$\therefore$ We have $B^2 = \{00, 01, 10, 11\}$

$e(00) = 00\ x_1\ x_2\ x_3$

$$[\,x_1\ x_2\ x_3\,] = [0\ \ 0]\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = [\,0\ 0\ 0\,]$$

$\Rightarrow$ $x_1 = x_2 = x_3 = 0$

$e(01) = 01\ x_1\ x_2\ x_3$

$$[\,x_1\ x_2\ x_3\,] = [0\ \ 1]\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = [\,1\ 0\ 1\,]$$

$\Rightarrow$ $x_1 = 1,\ x_2 = 0,\ x_3 = 1$

$e(10) = 10\ x_1\ x_2\ x_3$

$$[\,x_1\ x_2\ x_3\,] = [1\ \ 0]\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = [\,0\ 1\ 1\,]$$

$\Rightarrow$ $x_1 = 0,\ x_2 = 1,\ x_3 = 1$

$e(11) = 11\ x_1\ x_2\ x_3$

$$[\,x_1\ x_2\ x_3\,] = [1\ \ 1]\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = [\,1\ 1\ 0\,]$$

$\Rightarrow$ $x_1 = 1,\ x_2 = 1,\ x_3 = 0$

Hence, $e_H : B^2 \to B^5$ is defined as

$e(00) = 00000 = x_0$ $\quad$ $e(01) = 01101 = x_1$

$e(10) = 10011 = x_2$ $\quad$ $e(11) = 11110 = x_3$

(i) let $x_t = 10100$

$$|x_0 \oplus x_t| = |x_t| = 2$$

$$|x_1 \oplus x_t| = |11001| = 3$$

$$|x_2 \oplus x_t| = |00111| = 3$$

$$|x_3 \oplus x_t| = |01010| = 2$$

$\Rightarrow$ Minimum distance is not unique.

Note : If minimum distance for x_t is not unique, then we see on priority basis which one comes first.

The required nearer word to x_t

$d(x_t) = d(x_0) = 00$

$\therefore$ decode word for 10100 is 00

(ii) x_t = 01101

$|x_0 \oplus x_t| = |x_t| = 3$

$|x_1 \oplus x_t| = |00000| = 0$

$|x_2 \oplus x_t| = |11110| = 4$

$|x_3 \oplus x_t| = |10011| = 3$

d (x_t) = d (x_1) = 01

∴ decode word for 01101 is 01.

(iii) 11011 = x_t

$|x_0 \oplus x_t| = |x_t| = 4$

$|x_1 \oplus x_t| = |10110| = 3$

$|x_2 \oplus x_t| = |01000| = 1$

$|x_3 \oplus x_t| = |00101| = 2$

d (x_t) = d (x_2) = 10

∴ decode word for 11011 is 10.

Q.3(a) Define universal and Existential quantifiers. Transcribe the following into logical notation Let the universe of discourse be the real numbers: **[6]**

(i) There are positive values of x and y such that x, y > 0

(ii) For every value of x there is some value of y such that x, y = 1

(iii) There is a value of x such that if y is positive then x + y is negative

Ans.: **Universal Quantifiers:** Universal quantifiers is a type of quantifier, a logical constant which is interpreted as "given any" or "For all" universal (∀) – For all / For each.

Existential Quantifier: Existential quantifiers is a type of quantifier, a logical constant which is interpreted as "there exists", "there is at least one", or "for some" Existential (∃) = For some object.

i) $\exists$ x, y   x, y > 0

ii) $\forall$ x $\exists$ y   x, y = 1

iii) $\exists$ x, y $\rightarrow$ ~(x + y)

Q.3(b) Let f : R → R be a function defined by f (x) = 2x – 3. **[4]**
Prove that it is bijective hence find inverse.

Ans.: Here f : R → R is defined by f (x) = 2x – 3

(i) Injective :

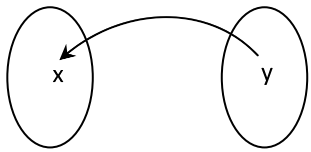
Consider $f(x_1) = f(x_2)$

$2x_1 - 3 = 2x_2 - 3$

$\Rightarrow$ $x_1 = x_2$

∴ f is injective.

(ii) Surjective :



R (domain) R (codomain)

Consider an arbitrary element y in R (codomain)

Let $y = f(x)$

$y = 2x - 3$

or $y + 3 = 2x$

or $x = \dfrac{y+3}{2}$

$\Rightarrow$ $\forall\, y \in$ R (codomain) $\exists$ pre image $x \in$ R (domain)

$\Rightarrow$ Range of f = codomain

$\Rightarrow$ f is surjective

$\because$ f is injective and surjective both

$\therefore$ f is bijective.

$\therefore$ f^{-1} exists.

$y = f(x) \Rightarrow x = f^{-1}(y)$

$y = 2x - 3$

$x = \dfrac{y+3}{2} = f^{-1}(y)$

$\therefore$ The rule for f^{-1} is

$$f^{-1}(x) = \frac{x+3}{2}$$

Q.3(c) A bag contains six white marbles and five red marbles. Find the number of ways four marbles can be drawn from the bag if (a) they can be any color; (b) two must be white and two red; (c) they must all be of the same color. **[4]**

Ans.: (a) The four marbles (of any color) can be chosen from the eleven marbles in

$$\binom{11}{4} = \frac{11.10.9.8}{1.2.3.4} = 330 \text{ ways}$$

(b) The two white marbles can be chosen in $\binom{6}{2}$ ways, and the two red marbles can be chosen in $\binom{5}{2}$ ways.

Thus there are $\binom{6}{2}\binom{5}{2} = \frac{6.5}{1.2}.\frac{5.4}{1.2} = 150$ ways of drawing two white marbles and two red marbles.

(c) There are $\binom{6}{4} = 15$ ways of drawing four white marbles, and $\binom{5}{4} = 5$ ways of drawing four red marbles. Thus there are 15 + 5 = 20 ways of drawing four marbles of the same colour.

Q.3(d) Prove that a set of non–zero real numbers forms an abelian group with respect to binary operation '$*$' where '$*$' is defined as $\forall$ a, b $\in$ R **[6]**

$$a * b = \frac{a.b}{2}$$

Ans.: (i) Closure axiom :

Let a, b $\in$ R

$\Rightarrow$ a. b $\in$ R, ½ $\in$ R $\Rightarrow$ $\frac{a.b}{2} \in R$

$\Rightarrow$ $a * b \in R$

$\Rightarrow$ 'R' is closed with respect to '$*$' operation

(ii) Associativity :

$\forall$ a, b, c $\in$ R

$$a * (b * c) = a * \frac{bc}{2} \quad \text{(by definition)}$$

$$= \frac{a\left(\frac{bc}{2}\right)}{2} = \frac{\left(\frac{ab}{2}\right)c}{2} = \left(\frac{ab}{2}\right) * c \quad \text{(by definition)}$$

$$= (a * b) * c$$

$\Rightarrow$ '$*$' is an associative operator

(iii) Identity element :

$\forall$ a $\in$ R $\exists$ an element e $\in$ R

such that

$$a * e = a$$

$$\frac{ae}{2} = a$$

$$e = 2 \in R$$

$\Rightarrow$ 2 is an identity element in 'R' with respect to '$*$'.

(iv) Inverses :

Let a $\in$ R $\exists$ $\alpha \in$ R

such that

$$a * \alpha = 2$$

$$\frac{a\alpha}{2} = 2$$

$$a\alpha = 4$$

$$\alpha = \frac{4}{a} \in R$$

$\Rightarrow$ All the elements are invertible $\because$ $0 \notin R$

(v) Commutativity :

$\forall$ a, b, $\in$ R

$$a * b = \frac{ab}{2} \quad \text{(by definition)}$$

$$= \frac{ba}{2}$$

$$= b * a \quad \text{(by definition)}$$

$\therefore$ $a * b = b * a$

$\because$ '$*$' is commutative with respect to R

$\Rightarrow$ '$*$' is commutative

$\Rightarrow$ (R, $*$) forms an abelian group.

Q.4(a) State the converse, inverse and contrapositive of the following : [4]

(i) If it is cold then he wears a hat

(ii) If an integer is a multiple of 2, then it is even.

Ans.: (i) p = it is cold

q = he wears a hat

~p = it is not cold

~q = he doesn't wears a hat

$p \to q$

Converse : $q \to p$

If he wears a hat, then it is cold.

Inverse : $\sim p \to \sim q$

If it is not cold then he doesn't wear a hat.

Contrapositive : $\sim q \to \sim p$

If he doesn't wear a hot then it is not cold.

(ii) p = an integer is a multiple of 2

q = it is even

~p = if an integer is not a multiple of 2

~q = it is not even

$p \to q$

Converse : $q \to p$

If it is even, then an integer is a multiple of 2.

Inverse : $\sim p \to \sim q$

If an integer is not a multiple of 2, then it is not even.

Contrapositive : $\sim q \to \sim p$

If it is not even, that an integer is not a multiple of 2.

Q.4(b) Consider the relation R on set of integers defined as xRy iff $y = x^k$; k is positive integer. Show that R is a partial order relation. [4]

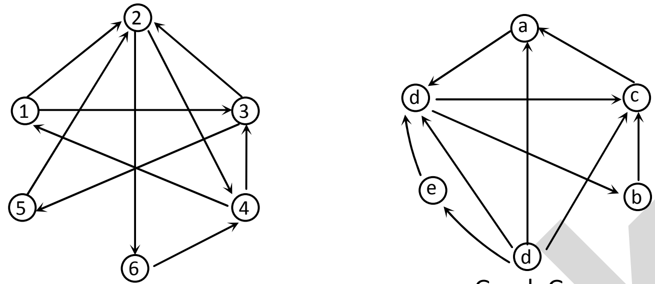
Ans.: R is relation defined as xRy iff $y = x^k$, where k is integer.

- Reflexive : For $x \in R$,
  $x = x^k$ iff k = 1
  $\therefore$ $x = (x)^1$ Hence, reflexive.

- Antisymmetric: For $x, y \in R$
  If $y = n^k$ then $x = y^k$
  iff y = x,
  $\therefore$ Hence, R is Antisymmetric.

- Transitive : For x, y, z $\leftarrow$ R
  If $y = x^k$ and $z = y^k$ then $z = x^k$
  Hence, R is transitive.
  Hence, R is a partial order relation.

Q.4(c) What are isomorphic graphs? Show that following two graphs are isomorphic. [6]



Graph G_1 Graph G_2

Ans.: (i) No. of vertices of G_1 = No. of vertices of G_2
6 = 6
(ii) No. of edges of G_1 = No. of edges of G_2
10 = 10

Degree of vertices of G_1	No. of Degree of vertices of G_2
deg(1) = 3	deg(a) = 3
deg(2) = 5	deg(b) = 2
deg(3) = 4	deg(c) = 4
deg(4) = 4	deg(d) = 5
deg(5) = 2	deg(m) = 2
deg(6) = 2	deg(p) = 4

$\therefore$ Graph G_1 and G_2 are isomorphic.

Q.4(d) Out of 250 candidates who failed in an examination, it was revealed that 128 failed in mathematics, 87 in physics and 134 in aggregate. 31 failed in mathematics and in Physics, 54 failed in the aggregate and in mathematics, 30 failed in the aggregate and in physics. Find how many candidates failed. [6]
(i) in all the three subjects.
(ii) in mathematics but not in physics.
(iii) in the aggregate but not in mathematics.
(iv) in physics but not in aggregate or in mathematics.

Ans.: Let M be the set of Candidates failed in Mathematics
P be the set of Candidates failed in Physics
A be the set of Candidates failed in Aggregate
$U = 250$, $|M| = 128$, $|P| = 87$, $|A| = 134$, $|M \cap P| = 31$, $|A \cap M| = 54$,
$|A \cap P| = 30$, $|A \cup P \cup M| = 250$

Using principle of Inclusion and Exclusion :
$|M \cup P \cup A| = |M| + |P| + |A| - |M \cap P| - |M \cap A| - |D \cap A| + |M \cap P \cap A|$
$250 = 128 + 87 + 134 - 31 - 54 - 30 + x$

$250 = 234 + x$

$\therefore \quad x = 16$

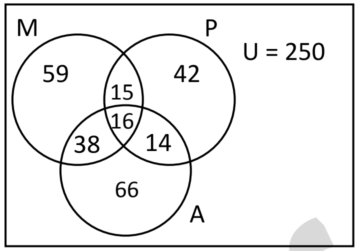
$\therefore \quad |M \cap P \cap A| = 16$

$\therefore$ 16 students have failed in all three subject

$\therefore$ 59 students have failed in Mathematics

$\therefore$ 42 students have failed in Physics

$\therefore$ 66 students have failed in Aggregate



Q.5(a) f : R → R is defined by f (x) = x^3 **[4]**
g : R → R is defined by g (x) = $4x^2 + 1$
h : R → R is defined by h (x) = 7x – 2
Find the rule defining : (i) fog, (ii) gof, (iii) (goh)of, (iv) go(hof)

Ans.: (i) $(fog)(x) = f\{g(x)\} = f(4x^2 + 1) = (4x^2 + 1)^3$

(ii) $(gof)(x) = g\{f(x)\} = g(x^3) = 4(x^3)^2 + 1$

Note : fog ≠ gof

(iii) $\{(goh)of\}(x) = [(goh)\{f(x)\}]$
$= (goh)(x^3) = g\{h(x^3)\} = g(7x^3 – 2)$
$= 4(7x^3 – 2)^2 + 1$

(iv) $\{go(hof)\}(x) = g\{(hof)(x)\}$
$= g[h\{f(x)\}] = g[h(x^3)] = g(7x^3 – 2)$
$= 4(7x^3 – 2)^2 + 1$

Note : (goh)of = go(hof)

Q.5(b) Show that in a group, ∀ a, b ∈ G, $(a * b)^2 = a^2 * b^2$, iff (G, *) must be abelian. **[4]**

Ans.: **If part :** ∀ a, b ∈ G, $(a * b)^2 = a^2 * b^2$ (given)

⇒ (a * b) * (a * b) = (a * a) * (b * b)

⇒ a * (b * a) * b = a * (a * b) * b (by associativity)

⇒ b * a = a * b by left & right cancellation law.

⇒ ∗ is commutative.

⇒ (G, *) is abelian group.

Only if part : Let (G, *) be an abelian group.

∴ a * b = b * a

We have to prove that

$(a * b)^2 = a^2 * b^2$

LHS = $(a * b)^2$
= (a * b) * (a * b)
= a * (b * a) * b … by associativity
= a * (a * b) * b … by commutativity
= (a * a) * (b * b) … by associativity
= $a^2 * b^2$

Q.5(c) Explain Warshwall's algorithm Let A = {1, 2, 3, 4, 5} and let R be a relation on A **[6]**
Such that R = {(1, 1), (1, 4), (2, 2), (3, 4), (3, 5), (4, 1), (5, 2), (5, 5)}
Find transitive closure of R by Warshwall's algorithm.

Ans.: A = {1, 2, 3, 4, 5}
R = {(1, 1), (1, 4), (2, 2), (3, 4), (3, 5), (4, 1), (5, 2), (5, 5)}

$$\therefore \quad M_R = w_0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

C_1 = 1's at 1, 4
R_1 = 1's at 1, 4

Put (1, 1), (4, 1), (1, 4), (4, 4)

$$\therefore \quad w_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

C_2 = 1's at 2, 5
R_2 = 1's at 2

Put 1's at , (2, 2), (5, 2)

$$w_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

C_3 = 1's at position
R_3 = 1's at position 4, 5
$w_3 = w_2$ $\quad C_4$ = 1's at 1, 3, 4
R_4 = 1's at 1, 4

$\therefore$ Put 1's at (1, 1), (1, 4), (3, 1), (3, 4), (4, 1), (4, 4)

$$\therefore \quad w_4 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

C_5 = 1's at 3, 5
R_5 = 1's at 2, 5

Put 1's at (3, 2), (3, 5), (5, 2)

$$w_5 = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix}\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \text{ which is a transitive closure of R.}$$
$$\quad 1 \quad 2 \quad 3 \quad 4 \quad 5$$

Q.5(d) Find the solution of recurrence relation : $a_r + 5a_{r-1} + 6a_{r-2} = 3r^2$ **[6]**

Ans.: $a_r + 5a_{r-1} + 6a_{r-2} = 3r^2$ ...(1)

This is a linear and non–homogeneous recurrence relation.

Its solution is given by

$$a_r = a_r^{(H)} + a_r^{(p)}$$

where $a_r^{(H)}$ is homogeneous solution and a_n^p is particular solution.

$x^2 + 5x + 6 = 0$

$\therefore$ $(x + 2)(x + 3) = 0$

$x = -2, x = -3 \Rightarrow a_r^{(H)} = u(-2)^r + v(-3)^r$ (u, v are constants)

For particular solution, the particular solution of $f(n) = 3r^2$ is given by $A_2r^2 + A_1r + A_0$ where A_2, A_1, A_0 are constants.

$\therefore$ $a_r^p = A_2r^2 + A_1r + A_0$

$\therefore$ $a_{r-1} = A_2(r-1)^2 + A_1(r-1) + A_0$

$\therefore$ $a_{r-2} = A_2(r-2)^2 + A_1(r-2) + A_0$

$\therefore$ $[A_2r^2 + A_1r + A_0] + 5[A_2(r-1) + A_1(r-1) + A_0]$
$+ 6[A_2(r-1)^2 + A_1(r-1) + A_0] = 3r^2$ ...(from (1))

Simplifying the above equation, we get

$$12A_2r^2 + (34A_2 - 12A_1)r + (12A_0 + 29A_2 - 17A_1) = 3r^2$$

Comparing coefficients, we get,

$$12A_2 = 3 \quad \Rightarrow \quad A_2 = \frac{1}{4}$$

$$-12A_1 + 34A_2 = 0 \quad \Rightarrow \quad A_1 = \frac{17}{24}$$

$$29A_2 - 17A_1 + 12A_0 = 0 \quad \Rightarrow A_0 = \frac{115}{288}$$

$\therefore$ Particular solution is, $a_n^p = \frac{1}{4}r^2 + \frac{17}{24}r + \frac{115}{288}$

Q.6(a) $f : R - \left\{\frac{2}{5}\right\} \rightarrow R - \left\{\frac{4}{5}\right\}$ defined by $f(x) = \frac{4x+3}{5x-2}$ show that the function is bijective and find rule for f^{-1}. **[4]**

Ans.: $\because$ $f : R - \left\{\frac{2}{5}\right\} \rightarrow R - \left\{\frac{4}{5}\right\}$ is defined by $f(x) = \frac{4x+3}{5x-2}$

(i) Injective :

Consider $f(x_1) = f(x_2)$

$$\frac{4x_1+3}{5x_1-2} = \frac{4x_2+3}{5x_2-2}$$

or $\quad (4x_1+3)(5x_2-2) = (4x_2+3)(5x_2-2)$

or $\quad 20x_1x_2 - 8x_1 + 15x_2 - 6 = 20x_1x_2 - 8x_2 + 15x_1 - 6$

or $\quad -8x_1 - 15x_1 = -8x_2 - 15x_2$

or $\quad -23x_1 = -23x_2$

or $\quad x_1 = x_2$

$\therefore$ f is injective.

(ii) Surjective :

Consider an arbitrary element y in R – {4/5} (codomain)

Let $\quad y = f(x)$

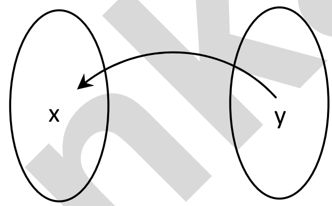
$$y = \frac{4x+3}{5x-2}$$

or $\quad 5xy - 4x = 4x + 3$

or $\quad 5xy - 4x = 2y + 3$

or $\quad x(5y - 4) = 2y + 3$

or $$x = \frac{2y+3}{5y-4}$$



– {2/5} (domain) R – {4/5} (codomain)

$\Rightarrow \quad \forall\, y \in R - \left\{\frac{4}{5}\right\}$ (codomain)

$\exists \quad$ pre image $x \in R - \left\{\frac{2}{5}\right\}$ (domain)

$\Rightarrow$ Range of f = codomain

$\Rightarrow$ f is surjective

$\because$ f is injective and surjective both

$\therefore$ f is bijective

$\therefore$ f^{-1} exist

$y = f(x) \Rightarrow x = f^{-1}(y) \qquad \Rightarrow x = \dfrac{2y+3}{5y-4} = f^{-1}(y)$

$\therefore$ The rule for f^{-1} is , $\quad f^{-1}(x) = \dfrac{2x+3}{5x-4}$

Q.6(b) How many friends must have to guarantee that atleast five of them will have birthday in the same month? [6]

Ans.: According to extended PHP one pigeonhole must contains atleast $\left[\frac{n-1}{m}\right]+1 = 5$ pigeonhole.

n = No. of pigeons = x

m = No. of pigeonholes = 12

$$\left[\frac{n-1}{m}\right]+1 = 5$$

$$\left\lceil \frac{x-1}{12} \right\rceil + 1 = 5$$

$$\left\lceil \frac{x-1}{12} \right\rceil = 4$$

x = 49

∴ 49 friends must be there so that atleast 5 of then must have birthday in a same month of year.

Q.6(c) A connected planar graph has 9 vertices having degrees 2,2,2,3,3,3,4,4 and 5. How many edges are there ? **[4]**

Ans.: By handshaking lemma

$$\sum_{i=1}^{n} d(v_i) = 2e$$

Where $d(v_i)$ = degree of i^{th} vertex

e = number of edges

∴ For given graph

$2 + 2 + 2 + 3 + 3 + 3 + 4 + 4 + 5 = 2.e$

$28 = 2e$

$e = 14$

∴ there are 14 edges.

Q.6(d) Let Z^+ is a set of positive integers and a relation R defined on Z^+ by a R b iff a | b then prove that R is a partial order relation and (Z^+ , |) is a poset. **[6]**

Ans.: (i) Reflexive :

∵ $a \mid a \quad \forall\ a \in Z^+$

∴ $aRa \quad \forall\ a \in Z^+$

⇒ | is reflexive.

(ii) Antisymmetric : Let aRb and bRa

⇒ a | b & b | a

⇒ $a = b \quad \forall\ a, b \in Z^+$

⇒ | is antisymmetric.

(iii) Transitive : Let aRb and bRc

⇒ a | b & b | c

⇒ a | c

⇒ aRc

⇒ | is transitive.

∴ | is partial order relation on Z^+.

∴ (Z^+, |) is a poset.

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