Applied Mathematics III

Prelim Paper Solution

Time : 3 Hrs.] **[Marks : 80**

Q.1(a) Find Laplace transform of te^{3t} cost. **[5]**

Ans.: $L\{\cos t\} = \frac{s}{s^2+1}$, $L\{t \cos t\} = (-1)\frac{d}{ds}\frac{s}{s^2+1}$

$\therefore$ $L\{t \cos t\} = -\left\{\frac{s^2+1-s(2s)}{(s^2+1)^2}\right\} = \frac{s^2-1}{(s^2+1)^2}$

by F.S.T.

$$L\left\{t\, e^{3t} \cos t\right\} = \frac{(s-3)^2-1}{\left[(s-3)^2+1\right]^2}$$

Q.1(b) Find $z\{2^k \cdot k^2\}$, $k \geq 0$. **[5]**

Ans.: $z\{1\} = \frac{z}{z-1}$

$$z\{k(1)\} = -z\frac{d}{dz}\frac{z}{z-1} = -z\left\{\frac{z-1-z}{(z-1)^2}\right\} = \frac{z}{(z-1)^2}$$

$$z\{k^2\} = z\{k \cdot k\} = -z\frac{d}{dz}\frac{z}{(z-1)^2} = -z\left\{\frac{(z-1)^2 - z2(z-1)}{(z-1)^4}\right\}$$

$$= \frac{-z(z-1)(z-1-2z)}{(z-1)^4} = \frac{z(z+1)}{(z-1)^3}$$

Now, $$z\{2^k k^2\} = \frac{\frac{z}{2}\left(\frac{z}{2}+1\right)}{\left(\frac{z}{2}-1\right)^3} = \frac{z(2z+4)}{(z-2)^3}$$

Q.1(c) Show that f(z) = sinh z is analytic. Hence find its derivative. **[5]**

Ans.: $f(z) = \sinh z = \sinh(x + iy)$

$= \sinh x \cosh iy + \cosh x \sinh iy$

$u + iv = \sinh x \cos y + i \cosh x \sin y$

$\Rightarrow$ $u = \sinh x \cos y$

$u_x = \cosh x \cos y$ ...(1)

$u_y = \sinh x \sin y$ ...(2)

$v = \cosh x \sin y$

$v_x = \sinh x \sin y$ ...(3)

$v_y = \cosh x \cos y$ ...(4)

From (1), (4) and (2), (3) $u_x = v_y$ $v_x = -u_y$

$\therefore$ f(z) = sinh z is analytic.

Now $F'(z) = u_x + iv_x\big|_{x=z, y=0} = \cosh x \cos y + i \sinh x \sin y\big|_{x=z, y=0}$

$f'(z) = \cosh z$

Q.1(d) Compute spearman's rank correlation for the data : **[5]**

X :	**18**	**20**	**34**	**52**	**12**
Y :	**39**	**23**	**35**	**18**	**46**

Ans.:

X	Y	R_x	R_y	$d = R_x - R_y$	d^2
18	39	4	2	2	4
20	23	3	4	–1	1
34	35	2	3	–1	1
52	18	1	5	–4	16
12	46	5	1	4	16

$\sum d^2 = 38$

$$R = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6(38)}{5(24)} \quad \Rightarrow \quad R = -0.9$$

Q.2(a) Show that the function $\omega = \frac{4}{z}$ transforms the straight line x = c in the z-plane into circle is w-plane. Find its centre and radius. **[6]**

Ans.: $x = c$ …(1)

Given $\omega = \frac{4}{z} \quad \Rightarrow z = \frac{4}{\omega} \quad \Rightarrow \quad x + iy = \frac{4}{u+iv}$

$\therefore \quad x + iy = \frac{4u}{u^2+v^2} - \frac{i4v}{u^2+v^2} \quad \Rightarrow \quad x = \frac{4u}{u^2+v^2}$

using this in (1) $\frac{4u}{u^2+v^2} = c \quad \Rightarrow u^2 + v^2 - \frac{4u}{c} = 0$

centre = $\left(\frac{z}{c}, 0\right)$, rad = $\frac{2}{c}$

Q.2(b) Show that $\int_0^\infty e^{-t}\int_0^t \frac{\sin u}{u}\,du\,dt = \frac{\pi}{4}$ **[6]**

Ans.: $\int_0^\infty e^{-t}\int_0^t \frac{\sin u}{u}\,du\,dt = L\left\{\int_0^t \frac{\sin u}{u}\,du\right\}\Bigg|_{s=1}$

Now

$$L\left\{\int_0^t \frac{\sin u}{u}\,du\right\} = \frac{1}{s}L\left\{\frac{\sin u}{u}\right\} = \frac{1}{s}\int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= \frac{1}{s}\left\{\tan^{-1}s\right\}_{s=s}^{s=\infty} = \frac{1}{s}\left[\frac{\pi}{2} - \tan^{-1}s\right]$$

$\therefore$ at s = 1,

$$= \frac{1}{1}\left[\frac{\pi}{2} - \tan^{-1}1\right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Q.2(c) **Obtain fourier series for f(x) =** $\begin{cases} 1+\dfrac{2x}{\pi} & -\pi \le x \le 0 \\ 1-\dfrac{2x}{\pi} & 0 \le x \le \pi \end{cases}$ **[8]**

Hence deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+....$

Ans.: f(x) is even function. $\Rightarrow$ bn = 0

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) \quad ...(1)$$

$$a_0 = \frac{1}{2\pi} 2\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)dx = \frac{1}{\pi}\left[x-\frac{x^2}{\pi}\right]_0^{\pi} \quad \Rightarrow a_0 = 0$$

$$a_n = \frac{1}{\pi}2\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos(nx)\,dx = \frac{2}{\pi}\left\{\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin(nx)}{n}\right)-\left(-\frac{2}{\pi}\right)\left(\frac{-\cos(nx)}{n^2}\right)\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\left[0-\frac{2}{\pi n^2}(-1)^n\right]-\left[0-\frac{2}{\pi n^2}\right]\right\}$$

$$a_n = \frac{4}{n^2\pi^2}\left[1-(-1)^n\right]$$

using this in (1)

$$f(x) = \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left[1-(-1)^n\right]\cos(nx)$$

for deduction put x = 0 and note that f(0) = 1

$$1 = \sum_{n=1}^{\infty}\frac{4}{\pi^2 n^2}\left[1-(-1)^n\right]$$

$$1 = \frac{4}{\pi^2}\left[\frac{2}{1^2}+0+\frac{2}{3^2}+0+\frac{2}{5^2}+...\right]$$

$$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...$$

Q.3(a) **Find inverse z-transform of** $\dfrac{z}{z-a}$ **in ROC** **[6]**

(i) |z| > a **(ii) |z| < a**

Ans.: $f(z) = \dfrac{z}{z-a}$

(i) $|z| > a$

$$\therefore \quad f(z) = \frac{z}{z\left(1-\frac{a}{z}\right)} = \left(1-\frac{a}{z}\right)^{-1} = 1+\frac{a}{z}+\frac{a^2}{z^2}+...+\frac{a^k}{z^k}+...$$

coefficient of z^{-k}, $k \ge 0$

$\Rightarrow x_k = a^k,\ k \ge 0$

(ii) $|z| < 0$

$$\therefore \quad f(z) = \frac{-z}{a\left(1-\frac{z}{a}\right)} = \frac{-z}{a}\left(1-\frac{z}{a}\right)^{-1} = \frac{-z}{a}\left\{1+\frac{z}{a}+\frac{z^2}{a^2}+\ldots+\frac{z^{k-1}}{a^{k-1}}+\ldots\right\}$$

$$f(z) = -\frac{z}{a}-\frac{z^2}{a^2}-\ldots-\frac{z^k}{a^k}-\ldots$$

coefficient of $z^k = -\frac{1}{a^k}$, $k \geq 1$

replacing k by $-$ k

coe. of $z^{-k} = -\frac{1}{a^{-k}}$, $-k \geq 1$

$\Rightarrow \quad x_k = -a^k, k \leq -1$

Q.3(b) For the lines of regression **[6]**

$6y - 5x = 90$, $15x - 8y = 130$ and $\sigma_x^2 = 16$

Find (i) $\bar{x}$, $\bar{y}$ (ii) r (iii) σy

Ans.: given $6y - 5x = 90$, $15x - 8y = 130$

solving we get $x = 30$, $y = 40$

i.e. $\bar{x} = 30$, $\bar{y} = 40$

$$y = \frac{5}{6}x+15 \quad \Rightarrow \quad b_{yx} = \frac{5}{6}$$

$$x = \frac{8}{15}y+\frac{130}{15} \quad \Rightarrow \quad b_{xy} = \frac{8}{15}$$

$$\therefore \quad r = \pm\sqrt{bxy\,byx} = +\sqrt{\frac{5}{6}\cdot\frac{8}{15}} \quad \Rightarrow \quad r = 0.67$$

given $\sigma_x^2 = 16 \quad \Rightarrow \quad \sigma_x = 4$

$$bxy = \frac{8}{15} \quad \Rightarrow \quad r\frac{\sigma_x}{\sigma_y} = \frac{8}{15} \quad \Rightarrow \quad \frac{2}{3}\left(\frac{4}{6y}\right) = \frac{8}{15}$$

$\sigma_y = 5$

Q.3(c) Solve the differential equation $\frac{dy}{dx}+2y+\int_0^t y\,dt$ = sin t using Laplace transform **[8]**

give y(0) = 1

Ans.:

$$L\{y'(t)\}+2L\{y(t)\}+L\left\{\int_0^t y(t)\,dt\right\} = L\{\sin t\}$$

$$5y(s)-1+2y(s)+\frac{1}{s}y(s) = \frac{1}{s^2+1}$$

$$\left(s+2+\frac{1}{s}\right)y(s) = \frac{1}{s^2+1}+1 = \frac{s^2+2}{s^2+1}$$

$$\frac{(s^2+2s+1)}{s}y(s) = \frac{s^2+2}{s^2+1} \Rightarrow y(s) = \frac{s(s^2+2)}{(s+1)^2(s^2+1)}$$

$$y(s) = \frac{a}{s+1} + \frac{b}{(s+1)^2} + \frac{cs+d}{s^2+1}, \; a = 1, \; b = -\frac{3}{2}, \; c = 0, \; d = \frac{1}{2}$$

$$\therefore \quad y(s) = \frac{1}{s+1} - \frac{3}{2}\frac{1}{(s+1)^2} + \frac{1}{2}\frac{1}{s^2+1}$$

taking inverse Laplace,

$$y(t) = e^{-t} - \frac{3}{2}te^{-t} + \frac{1}{2}\sin t$$

Q.4(a) Find orthogonal trajectory to the family of curves $e^{-x}\cos y + xy$ = constant in X – Y plane. **[6]**

Ans.: If f(z) = u + iv analytic then v is orthogonal to u

Let $u = e^{-x}\cos y + xy$

$$dv = v_x\, d_x + v_y\, d_y = -u_y\, dx + u_x\, d_y \qquad [\because\; u_x = v_y,\; v_x = -u_y]$$

$$dv = -\left(-e^{-x}\sin y + x\right) + \left(-e^{-x}\cos y + y\right) dy$$

which is exact D.E.

integrating

$$v = \int_{y=\text{cont}} \left(e^{-x}\sin y + x\right) dx + \int_{\text{free from } x} \left(-e^{-x}\cos y + y\right) dy + c$$

$$v = -e^{-x}\sin y + \left(\frac{x^2}{2}\right) + \left(\frac{y^2}{2}\right) + c$$

Q.4(b) Show that $\cos x = 8\pi \sum_{m=1}^{\infty} \frac{m}{4m^2-1}\sin(2mx)$, If $0 < x < \pi$ **[6]**

Ans.: Half range sine series $f(x) = \sum_{n=1}^{\infty} bn\sin(nx)$ ...(1)

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cos x \sin(nx)\, dx = \frac{2}{\pi}\int_0^{\pi}\left[\frac{\sin(nx+x)+\sin(nx-x)}{2}\right]dx$$

$$= \frac{1}{\pi}\left\{\frac{-\cos(nx+x)}{n+1} - \frac{\cos(nx-x)}{n-1}\right\}_0^{\pi}$$

$$= \frac{1}{\pi}\left\{\left[\frac{(-1)^n}{n+1} + \frac{(-1)^n}{n-1}\right] - \left[-\frac{1}{n+1} - \frac{1}{n-1}\right]\right\}$$

$$b_n = \frac{1}{\pi}\left[1+(-1)^n\right]\left[\frac{1}{n+1} + \frac{1}{n-1}\right] = \frac{2n}{\pi(n^2-1)}\left[1+(-1)^n\right]$$

$n \neq 1$

$$\text{Now } b_1 = \frac{2}{\pi}\int_0^{\pi}\sin x\cos x\, dx = \frac{1}{\pi}\int_0^{\pi}\sin 2x\, dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi} = -\frac{1}{2\pi}[1-1] \qquad \Rightarrow b_1 = 0$$

$$\Rightarrow \cos x = \sum_{n=2}^{\infty}\frac{2n}{\pi(n^2-1)}\left[1+(-1)^n\right]\sin(nx)$$

put n = 2m

$$\therefore \quad \cos x = \sum_{2m=2}^{\infty} \frac{2(2m)}{\pi\left(4m^2-1\right)}\left[1+(-1)^{2m}\right]\sin(2mx)$$

$$\therefore \quad \cos x = \frac{8}{\pi}\sum_{m=1}^{\infty}\frac{m}{4m^2-1}\sin(2mx)$$

Q.4(c) Find Bilinear transformation which maps the points 1, i, –1 onto the points i, 0, –i. Hence find fixed points and image of |z| < 1. **[8]**

Ans.: Consider B. I. $\omega = \dfrac{az+b}{cz+d}$ ...(1)

given $z = 1, \omega = i$ (1)

$\Rightarrow i = \dfrac{a+b}{c+d} \quad \Rightarrow \quad ic + id = a + b$ ...(2)

given $z = i, \omega = 0$ (1)

$\Rightarrow 0 = \dfrac{ia+b}{ic+d} \quad \Rightarrow b = -ia$ ...(3)

given $z = -1, \omega = -i$ (1)

$\Rightarrow -i = \dfrac{-a+b}{-c+d} \quad \Rightarrow \quad ic - id = -a + b$ ...(4)

(2) + (4) $\Rightarrow 2ic = 2b \quad \Rightarrow \quad 2ic = -2\,ia \quad \Rightarrow \quad c = -a$ ...(5)

(2) – (4) $\Rightarrow 2id = 2a \quad \Rightarrow \quad d = -ia$ ...(6)

using (3), (5), (6), in (1)

$\omega = \dfrac{az-ia}{-az-ia} \quad \Rightarrow \quad \omega = \dfrac{i-z}{i+z}$ ...(7)

For fixed points w = z, (7) $\Rightarrow z = \dfrac{i-z}{i+z}$

$z^2 + iz = i - z \quad \Rightarrow \quad z^2 + (i + 1)z - i = 0$

$$z = \frac{-i-1\pm\sqrt{-1+1+2i+4i}}{2}$$

$$= \frac{-i-1\pm\sqrt{6i}}{2}$$

to find image of |z| < 1 : (7) $\Rightarrow \quad i\omega + \omega z = i - z \Rightarrow (1 + \omega)z = i - i\omega$

$$z = \frac{i-i(u+iv)}{1+u+iv}$$

$\therefore \quad |z| < 1 \quad \Rightarrow \quad \left|\dfrac{i-iu+v}{1+u+iv}\right| < 1$

$|i - iu + v| = |1 + u + iv|$

$\sqrt{v^2+(1-u)^2} < \sqrt{(1+u)^2+v^2}$

$\Rightarrow \quad v^2 + 1 - 2u + u^2 < 1 + 2u + u^2 + v^2 \Rightarrow 0 < 4u$

$\Rightarrow \quad 0 < u$

Q.5(a) **Find z – transform of** $X_k = \begin{cases} 3^k & k < 0 \\ 2^k & k \geq 0 \end{cases}$ **[6]**

Ans.: by definition $z\{x_k\} = \sum_{k=-\infty}^{\infty} x_k\, z^{-k} \quad = \sum_{k=-1}^{-\infty} 3^k z^{-k} + \sum_{k=0}^{\infty} 2^k z^{-k}$

$$= \left\{\frac{z}{3} + \frac{z^2}{3^2} + \ldots\right\} + \left\{1 + \frac{2}{z} + \frac{2^2}{z^2} + \ldots\right\} \qquad \text{G.P.}$$

$$= \frac{\frac{z}{3}}{1 - \frac{z}{3}} + \frac{1}{1 - \frac{2}{z}} \quad \Rightarrow \quad z\{x_k\} = \frac{z}{3-z} + \frac{z}{z-2}$$

For ROC, $\left|\frac{z}{3}\right| < 1$ and $\left|\frac{2}{z}\right| < 1$

$\Rightarrow$ $|z| < 3$ and $2 < |z|$

$\Rightarrow$ $2 < |z| < 3$

Q.5(b) **If $f(x) = c_1\,\phi_1(x) + c_2\,\phi_2(x) + c_3\,\phi_3(x)$, where c_1, c_2, c_3 are constant and ϕ_1, ϕ_2, ϕ_3 are orthonormal functions, on the set (a, b).** **[6]**

Show that $\int_a^b [f(x)]^2\,dx = c_1^2 + c_2^2 + c_3^2$

Ans.: Given ϕ_1, ϕ_2 ϕ_3 are orthonormal on (a, b)

$$\Rightarrow \int_a^b \phi_1^2 dx = \int_a^b \phi_2^2\,dx = \int_a^b \phi_3^2\,dx = 1 \qquad \ldots(1)$$

$$\int_a^b \phi_1\phi_2\,dx = \int_a^b \phi_1\,\phi_3\,dx = \int_a^b \phi_2\phi_3\,dx = 0 \qquad \ldots(2)$$

Now

$f = c_1\,\phi_1 + c_2\,\phi_2 + c_3\,\phi_3$ squaring

$f^2 = c_1^2\,\phi_1^2 + c_2^2\,\phi_2^2 + c_3^2\phi_3^2 + 2c_1c_2\phi_1\phi_2 + 2c_1c_3\phi_1\phi_3 + 2c_2c_3\phi_2\phi_3$

$$\therefore \int_a^b f^2 dx = c_1^2\int_a^b \phi_1^2\,dx + c_2^2\int_a^b \phi_2^2\,dx + c_3^2\int_a^b \phi_3^2\,dx$$

$$+2c_1c_2\int_a^b \phi_1\phi_2 dx + 2c_1c_3\int_a^b \phi_1\phi_3 dx + 2c_2c_3\int_a^b \phi_2\phi_3\,dx$$

using equation (1), (2) we get

$$\int_a^b [f(x)]^2\,dx = c_1^2 + c_2^2 + c_3^2$$

Q.5(c) **Find inverse Laplace Transform of** **[8]**

(i) $\log\left(1 + \frac{a^2}{s^2}\right)$ **(ii)** $\frac{e^{-s}}{s^2 + s + 1}$

Ans.: (i) Let $f(s) = \log\left(1+\frac{a^2}{s^2}\right) = \log\left(\frac{s^2+a^2}{s^2}\right)$

$$f(s) = \log(s^2+a^2) - \log s^2$$

$$\Rightarrow f'(s) = \frac{2s}{s^2+a^2} - \frac{2}{s}$$

taking L^{-1} we get

$$L^{-1}\{f'(s)\} = 2\cos at - 2$$

$$\Rightarrow -t\,f(t) = 2\cos at - 2$$

$$\Rightarrow f(t) = \frac{2(1-\cos at)}{t}$$

(ii) Let $f(s) = \frac{1}{s^2+s+1} = \frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$

$$\therefore L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\}$$

$$\Rightarrow F(t) = e^{-\frac{1}{2}t} L^{-1}\left\{\frac{1}{s^2+\frac{3}{4}}\right\} = \frac{e^{-\frac{1}{2}t}}{\frac{\sqrt{3}}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

we know that,

$$L^{-1}\{e^{-as} f(s)\} = f(t-a)\,H(t-a)$$

$$\therefore L^{-1}\left\{\frac{e^{-s}}{s^2+s+1}\right\} = \frac{2}{\sqrt{3}} e^{-\frac{1}{2}(t-1)} \sin\left[\frac{\sqrt{3}}{2}(t-1)\right]\cdot H(t-1)$$

Q.6(a) Obtain complex form of fourier series for F(x) = $e^{\alpha x}$, in $(-\pi, \pi)$ where a is not an integer. [6]

Ans.: Complex form of fourier series is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad \ldots(1)$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax} e^{-inx}\,dx = \frac{1}{2\pi}\left\{\frac{e^{ax-inx}}{a-in}\right\}_{-\pi}^{\pi} = \frac{1}{2\pi}\frac{(a+in)}{(a+in)(a-in)}\left\{e^{a\pi}e^{-in\pi} - e^{-a\pi}e^{in\pi}\right\}$$

$$= \frac{1}{2\pi}\frac{(a+in)}{(a^2+n^2)}(-1)^n\left(e^{a\pi} - e^{-a\pi}\right)$$

$$c_n = \frac{(a+in)}{\pi\left(a^2+n^2\right)}(-1)^n \sinh(a\pi)$$

using this in (1)

$$e^{ax} = \sum_{n=-\infty}^{\infty}\frac{(a+in)}{\pi\left(a^2+n^2\right)}(-1)^n \sinh(a\pi)\, e^{inx}$$

Q.6(b) Fit a curve $y = a\cdot b^x$ to the following data, using method of least squares. **[6]**

X :	2	3	4	5	6
Y :	144	172.8	207.4	248.8	298.5

Ans.: $y = a\cdot b^x$ (taking log) ...(1)

$\log y = \log a + x \log b$

$\Rightarrow \sum \ln g = n \ln a + l\ nb \sum x$ ...(2)

$\sum x \ln y = \ln a \sum x + \ln b \sum x^2$ ...(3)

(2) $\Rightarrow$ 26.58 = 5A + 20 B

(3) $\Rightarrow$ 108.51 = 20A + 90 B

} where A = ln a, B = ln b

Solving

B = 0.179

$\therefore$ (1) $\Rightarrow$

$y = (101.49)(1.196)^x$

ln a = 4.62

a = 101.49

ln b = 0.179

b = 1.196

Q.6(c) Find imaginary part of analytic function whose real part is $e^{2x}(x \cos 2y - y \sin 2y)$, Also verify that v is harmonic function. **[8]**

Ans.: Given $u = e^{2x}(x \cos 2y - y \sin 2y)$

$f(z) = u + iv$ analytic $\Rightarrow u_x = v_y,\ v_x = -u_y$ } $\rightarrow$ (1)

$dv = v_x dx + v_y dy = -u_y dx + u_x dy$ [$\because$ (1)]

$$dv = -e^{2x}(-2x \sin 2y - \sin 2y - 2y \cos 2y)\,dx + \{e^{2x}(\cos 2y) + 2e^{2x}(x \cos 2y - y \sin 2y)dy\}$$

which is exact.

by integration,

$$v = \int_{y = \text{cont}} e^{2x}(2x \sin 2y + \sin 2y + 2y \cos 2y)\,dx + \int_{\text{free from } x} 0dy + c$$

$$= (2x \sin 2y + \sin 2y + 2y \cos 2y)\left(\frac{e^{2x}}{2}\right) - (2 \sin 2y)\left(\frac{e^{2x}}{4}\right) + c$$

$$= \frac{e^{2x}}{2}\{2x \sin 2y + \sin 2y + 2y \cos 2y - \sin 2y\}$$

$$v = e^{2x}(x \sin 2y + y \cos 2y)$$

$$v_x = 2e^{2x}(x \sin 2y + y \cos 2y) + e^{2x} \sin 2y$$

$$v_{xx} = 4e^{2x}(x \sin 2y + y \cos 2y) + 2e^{2x} \sin 2y + 2e^{2x} \sin 2y$$

$$v_{xx} = 4xe^{2x} \sin 2y + 4y\,e^{2x} \cos 2y + 4\,e^{2x} \sin 2y \quad ...(2)$$

$$v_y = e^{2x}(2x \cos 2y + \cos 2y - 2y \sin 2y)$$

$$v_{yy} = e^{2x}(-4x \sin 2y - 2 \sin 2y - 2 \sin 2y - 4y \cos 2y)$$

$$v_{yy} = -4xe^{2x} \sin 2y - 4e^{2x} \sin 2y - 4y\,e^{2x} \cos 2y \quad ...(3)$$

adding (2), (3) $V_{xx} + v_{yy} = 0$

$\Rightarrow$ v is harmonic.

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