# Applied Mathematics - III

**Time : 3 Hrs.]** Prelim Paper Solution **[Marks : 80**

**Q.1(a) Find Laplace transform of $te^{3t}$ cost.** **[5]**

**Ans.:** $L\{\cos t\} = \frac{s}{s^2+1}$ , $L\{t\cos t\} = (-1)\frac{d}{ds}\frac{s}{s^2+1}$

$\therefore$ $L\{t\cos t\} = -\left\{\frac{s^2+1-s(2s)}{(s^2+1)^2}\right\} = \frac{s^2-1}{(s^2+1)^2}$

by F.S.T.

$$L\left\{t\,e^{3t}\cos t\right\} = \frac{(s-3)^2-1}{\left[(s-3)^2+1\right]^2}$$

**Q.1(b) Prove that : $\overline{f}$ = (x + 2y + az)i + (bx – 3y – z)j + (4x + cy + 2z)k is solenoidal and determine the constants a, b, c if $\overline{f}$ is irrotational.** **[5]**

**Ans.:** $\nabla\cdot\overline{f}$ **= 1 – 3 + 2 = 0** $\Rightarrow$ $\overline{f}$ **is solenoidal.**

given $\nabla\times\overline{f} = 0$

$$\Rightarrow \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = 0$$

(c + 1)i – (4 – a)j + (b – 2) k = 0i + 0j + 0k

$\Rightarrow$ c = –1, a = 4, b = 2

**Q.1(c) Show that f(z) = sinh z is analytic. Hence find its derivative.** **[5]**

**Ans.:** f(z) = sinh z = sinh(x + iy)

= sinh x cosh iy + cosh x sinh iy

u + iv = sinh x cosy + i cosh x siny

$\Rightarrow$ u = sinh x cosy

$u_x$ = cosh x cosy ...(1)

$u_y$ = sinh x siny ...(2)

v = cosh x siny

$v_x$ = sinh x siny ...(3)

$v_y$ = cosh x cosy ...(4)

From (1), (4) and (2), (3) $u_x = v_y$ $v_x = -u_y$

$\therefore$ f(z) = sinh z is analytic.

Now $F'(z) = u_x + iv_x\big|_{x=z,\,y=0}$

$= \cosh x\cos y + i\sinh x\sin y\big|_{x=z,\,y=0}$

f′(z) = cosh z

**Q.1(d)** **Prove that $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cdot \cos x$** **[5]**

**Ans.:** We know that $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

$$\Rightarrow \quad \frac{d}{dx} x^{\frac{1}{2}} J_{\frac{1}{2}}(x) = x^{\frac{1}{2}} J_{-\frac{1}{2}}$$

$$\Rightarrow \quad \frac{d}{dx}\left[x^{\frac{1}{2}} \sqrt{\frac{2}{\pi x}} \sin x\right] = x^{\frac{1}{2}} J_{-\frac{1}{2}}$$

$$\sqrt{\frac{2}{\pi}} \frac{d}{dx} \sin x = x^{\frac{1}{2}} J_{-\frac{1}{2}} \Rightarrow \sqrt{\frac{2}{\pi}} \cos x = x^{\frac{1}{2}} J_{-\frac{1}{2}}$$

$$\Rightarrow \quad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

**Q.2(a)** **Show that the function $\omega = \frac{4}{z}$ transforms the straight line x = c in the z-plane into circle is w-plane. Find its centre and radius.** **[6]**

**Ans.:** $x = c$ …(1)

Given $\omega = \frac{4}{z} \Rightarrow z = \frac{4}{\omega} \Rightarrow x + iy = \frac{4}{u+iv}$

$$\therefore \quad x + iy = \frac{4u}{u^2+v^2} - \frac{i4v}{u^2+v^2} \Rightarrow x = \frac{4u}{u^2+v^2}$$

using this in (1) $\frac{4u}{u^2+v^2} = c \Rightarrow u^2 + v^2 - \frac{4u}{c} = 0$

centre = $\left(\frac{z}{c}, 0\right)$, rad = $\frac{2}{c}$

**Q.2(b)** **Show that $\int_0^\infty e^{-t} \int_0^t \frac{\sin u}{u} du\, dt = \frac{\pi}{4}$** **[6]**

**Ans.:** $\int_0^\infty e^{-t} \int_0^t \frac{\sin u}{u} du\, dt = L\left\{\int_0^t \frac{\sin u}{u} du\right\}\Bigg|_{s=1}$

Now

$$L\left\{\int_0^t \frac{\sin u}{u} du\right\} = \frac{1}{s} L\left\{\frac{\sin u}{u}\right\} = \frac{1}{s}\int_s^\infty \frac{1}{s^2+1} ds$$

$$= \frac{1}{s}\left\{\tan^{-1} s\right\}_{s=s}^{s=\infty} = \frac{1}{s}\left[\frac{\pi}{2} - \tan^{-1} s\right]$$

$\therefore$ at s = 1,

$$= \frac{1}{1}\left[\frac{\pi}{2} - \tan^{-1} 1\right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

**Q.2(c)** **Obtain fourier series for f(x) =** $\begin{cases} 1+\dfrac{2x}{\pi} & -\pi \le x \le 0 \\ 1-\dfrac{2x}{\pi} & 0 \le x \le \pi \end{cases}$ **[8]**

**Hence deduce that** $\dfrac{\pi^2}{8} = \dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+....$

**Ans.:** f(x) is even function. $\Rightarrow$ bn = 0

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) \qquad ...(1)$$

$$a_0 = \frac{1}{2\pi}2\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)dx = \frac{1}{\pi}\left[x-\frac{x^2}{\pi}\right]_0^{\pi} \qquad \Rightarrow a_0 = 0$$

$$a_n = \frac{1}{\pi}2\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos(nx)\,dx = \frac{2}{\pi}\left\{\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin(nx)}{n}\right)-\left(-\frac{2}{\pi}\right)\left(\frac{-\cos(nx)}{n^2}\right)\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\left[0-\frac{2}{\pi n^2}(-1)^n\right]-\left[0-\frac{2}{\pi n^2}\right]\right\}$$

$$a_n = \frac{4}{n^2\pi^2}\left[1-(-1)^n\right]$$

using this in (1)

$$f(x) = \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left[1-(-1)^n\right]\cos(nx)$$

for deduction put x = 0 and note that f(0) = 1

$$1 = \sum_{n=1}^{\infty}\frac{4}{\pi^2 n^2}\left[1-(-1)^n\right]$$

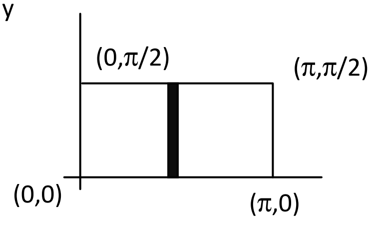
$$1 = \frac{4}{\pi^2}\left[\frac{2}{1^2}+0+\frac{2}{3^2}+0+\frac{2}{5^2}+...\right] \qquad \Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...$$

**Q.3(a)** **Evaluate by Green's Theorem** $\int_C \left(e^{-x}\sin y\,dx + e^{-x}\cos y\,dy\right)$ **[6]**

**where C is rectangle with vertices (0, 0), (π, 0), (π,** $\frac{\pi}{2}$**), (0,** $\frac{\pi}{2}$ **)**

**Ans.:** $\int P dx + Q dy = \iint (Q_x - P_y)\,dxdy$, **P =** $e^{-x}$ **sin y, Q =** $e^{-x}$ **cos y**

$$= \iint\left(-e^{-x}\cos y - e^{-x}\cos y\right)dx\,dy$$




$$= -2\int_0^{\pi}\left[\int_0^{\frac{\pi}{2}} e^{-x}\cos y\,dy\right]dx$$

$$= -2\int_0^{\pi} e^{-x}\left(\sin y\right)_0^{\frac{\pi}{2}}\,dx$$

$$= -2\left(\frac{e^{-x}}{-1}\right)_0^{\pi} \qquad = 2(e^{-\pi}-1)$$

**Q.3(b)** **Prove that $J_2'(x) = \left(1-\frac{4}{x^2}\right)J_1(x) + \frac{2}{x}J_0(x)$** **[6]**

**Ans.:** We know that

$$xJ'_n = -nJ_n + xJ_{n-1}$$

put n = 2, $xJ'_2 = -2J_2 + xJ_1 \quad \Rightarrow \quad J_2' = -\frac{2}{x}J_2 + J_1 \quad \rightarrow (1)$

by recurrence formula, $J'_n = \frac{n}{x}J_n - J_{n+1}$

also $J'_n = \frac{-n}{x}J_n + J_{n-1}$

equating

$$\frac{n}{x}J_n - J_{n+1} = \frac{-n}{x}J_n + J_{n-1}$$

$\Rightarrow \quad J_{n+1} = \frac{2n}{x}J_n - J_{n-1}$

put n = 1, $J_2 = \frac{2}{x}J_1 - J_0$ using this in (1)

$$J'_2 = -\frac{2}{x}\left[\frac{2}{x}J_1 - J_0\right] + J_1$$

$$J'_2 = \left(\frac{-4}{x^2}+1\right)J_1 + \frac{2}{x}J_0$$

**Q.3(c)** **Solve the differential equation $\frac{dy}{dx} + 2y + \int_0^t y\,dt = \sin t$ using Laplace transform give y(0) = 1** **[8]**

**Ans.:**

$$L\{y'(t)\} + 2L\{y(t)\} + L\left\{\int_0^t y(t)\,dt\right\} = L\{\sin t\}$$

$$5y(s) - 1 + 2y(s) + \frac{1}{s}y(s) = \frac{1}{s^2+1}$$

$$\left(s+2+\frac{1}{s}\right)y(s) = \frac{1}{s^2+1} + 1 = \frac{s^2+2}{s^2+1}$$

$$\frac{(s^2+2s+1)}{s}y(s) = \frac{s^2+2}{s^2+1} \Rightarrow y(s) = \frac{s(s^2+2)}{(s+1)^2(s^2+1)}$$

$$y(s) = \frac{a}{s+1} + \frac{b}{(s+1)^2} + \frac{cs+d}{s^2+1},\ a = 1,\ b = -\frac{3}{2},\ c = 0,\ d = \frac{1}{2}$$

$$\therefore \quad y(s) = \frac{1}{s+1} - \frac{3}{2}\frac{1}{(s+1)^2} + \frac{1}{2}\frac{1}{s^2+1}$$

taking inverse Laplace,

$$y(t) = e^{-t} - \frac{3}{2}te^{-t} + \frac{1}{2}\sin t$$

**Q.4(a)** **Find orthogonal trajectory to the family of curves $e^{-x}\cos y + xy$ = constant in X – Y plane.** **[6]**

**Ans.:** If f(z) = u + iv analytic then v is orthogonal to u

Let $u = e^{-x}\cos y + xy$

$$dv = v_x\, d_x + v_y\, d_y = -u_y\, dx + u_x\, d_y \qquad [\because\ u_x = v_y,\ v_x = -u_y]$$

$$dv = -\left(-e^{-x}\sin y + x\right) + \left(-e^{-x}\cos y + y\right) dy$$

which is exact D.E.

integrating

$$v = \int_{y=\text{cont}} \left(e^{-x}\sin y + x\right) dx + \int_{\text{free from } x} \left(-e^{-x}\cos y + y\right) dy + c$$

$$v = -e^{-x}\sin y + \left(\frac{x^2}{2}\right) + \left(\frac{y^2}{2}\right) + c$$

**Q.4(b)** **Show that $\cos x = 8\pi \sum_{m=1}^{\infty} \frac{m}{4m^2 - 1}\sin(2mx)$, If $0 < x < \pi$** **[6]**

**Ans.:** Half range sine series $f(x) = \sum_{n=1}^{\infty} bn\sin(nx)$ ...(1)

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cos x \sin(nx)\, dx = \frac{2}{\pi}\int_0^{\pi}\left[\frac{\sin(nx+x) + \sin(nx-x)}{2}\right] dx$$

$$= \frac{1}{\pi}\left\{\frac{-\cos(nx+x)}{n+1} - \frac{\cos(nx-x)}{n-1}\right\}_0^{\pi}$$

$$= \frac{1}{\pi}\left\{\left[\frac{(-1)^n}{n+1} + \frac{(-1)^n}{n-1}\right] - \left[-\frac{1}{n+1} - \frac{1}{n-1}\right]\right\}$$

$$b_n = \frac{1}{\pi}\left[1 + (-1)^n\right]\left[\frac{1}{n+1} + \frac{1}{n-1}\right] = \frac{2n}{\pi(n^2-1)}\left[1 + (-1)^n\right]$$

$n \neq 1$

$$\text{Now } b_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x\, dx = \frac{1}{\pi}\int_0^{\pi} \sin 2x\, dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi} = -\frac{1}{2\pi}[1-1] \qquad \Rightarrow \quad b_1 = 0$$

$$\Rightarrow \quad \cos x = \sum_{n=2}^{\infty} \frac{2n}{\pi(n^2-1)}\left[1 + (-1)^n\right]\sin(nx)$$

put n = 2m

$$\therefore \quad \cos x = \sum_{2m=2}^{\infty} \frac{2(2m)}{\pi(4m^2-1)}\left[1 + (-1)^{2m}\right]\sin(2mx)$$

$$\therefore \quad \cos x = \frac{8}{\pi}\sum_{m=1}^{\infty} \frac{m}{4m^2-1}\sin(2mx)$$

**Q.4(c)** **Find Bilinear transformation which maps the points 1, i, –1 onto the points i, 0, –i. Hence find fixed points and image of |z| < 1.** **[8]**

**Ans.:** Consider B. I. $\omega = \dfrac{az+b}{cz+d}$ ...(1)

given $z = 1, \omega = i$ (1)

$\Rightarrow\ i = \dfrac{a+b}{c+d} \quad \Rightarrow\ ic + id = a + b$ ...(2)

given $z = i, \omega = 0$ (1)

$\Rightarrow\ 0 = \dfrac{ia+b}{ic+d} \quad \Rightarrow b = -ia$ ...(3)

given $z = -1, \omega = -i$ (1)

$\Rightarrow\ -i = \dfrac{-a+b}{-c+d} \quad \Rightarrow\ ic - id = -a + b$ ...(4)

(2) + (4) $\Rightarrow 2ic = 2b \quad \Rightarrow\ 2ic = -2\,ia \quad \Rightarrow\ c = -a$ ...(5)

(2) – (4) $\Rightarrow 2id = 2a \quad \Rightarrow\ d = -ia$ ...(6)

using (3), (5), (6), in (1)

$\omega = \dfrac{az-ia}{-az-ia} \quad \Rightarrow\ \omega = \dfrac{i-z}{i+z}$ ...(7)

For fixed points w = z, (7) $\Rightarrow\ z = \dfrac{i-z}{i+z}$

$z^2 + iz = i - z \quad \Rightarrow\ z^2 + (i+1)\,z - i = 0$

$z = \dfrac{-i-1\pm\sqrt{-1+1+2i+4i}}{2} = \dfrac{-i-1\pm\sqrt{6i}}{2}$

to find image of |z| < 1 : (7) $\Rightarrow\ i\omega + \omega z = i - z \Rightarrow (1+\omega)z = i - i\omega$

$z = \dfrac{i-i(u+iv)}{1+u+iv}$

$\therefore\ |z| < 1 \ \Rightarrow\ \left|\dfrac{i-iu+v}{1+u+iv}\right| < 1$

$|i - iu + v| = |1 + u + iv|$

$\sqrt{v^2+(1-u)^2} < \sqrt{(1+u)^2+v^2}$

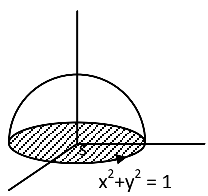
$\Rightarrow\ v^2 + 1 - 2u + u^2 < 1 + 2u + u^2 + v^2 \ \Rightarrow 0 < 4u$

$\Rightarrow\ 0 < u$

**Q.5(a)** **Use Stoke's Theorem to evaluate $\int_c \overline{f}\cdot\overline{dr}$** **[6]**

**Where $\overline{f} = yi + zj + xk$ and c is boundary of the surface $x^2 + y^2 = 1 - z$, $z > 0$.**

**Ans.:** $\nabla\times\overline{f} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$




$\nabla\times\overline{f} = -i - j - k, \qquad n'' = k = 0i + 0j + k$

$ds = dx\,dy$

$\therefore \quad \int_c \bar{f} \cdot d\bar{r} = \iint \left(\nabla \times \bar{f}\right) \cdot n'' ds \quad = \iint -dx\,dy = -\text{Area of circle}$

$= -\pi(1)^2 = -\pi$

**Q.5(b)** **If f(x) = $c_1\,\phi_1(x) + c_2\,\phi_2(x) + c_3\,\phi_3(x)$, where $c_1, c_2, c_3$ are constant and $\phi_1, \phi_2, \phi_3$ are orthonormal functions, on the set (a, b).** **[6]**

**Show that** $\int_a^b [f(x)]^2\,dx = c_1^2 + c_2^2 + c_3^2$

**Ans.:** Given $\phi_1, \phi_2\, \phi_3$ are orthonormal on (a, b)

$\Rightarrow \int_a^b \phi_1^2 dx = \int_a^b \phi_2^2\,dx = \int_a^b \phi_3^2\,dx = 1$ ...(1)

$\int_a^b \phi_1\phi_2\,dx = \int_a^b \phi_1\,\phi_3\,dx = \int_a^b \phi_2\phi_3\,dx = 0$ ...(2)

Now

$f = c_1\,\phi_1 + c_2\,\phi_2 + c_3\,\phi_3$ squaring

$f^2 = c_1^2\,\phi_1^2 + c_2^2\,\phi_2^2 + c_3^2\phi_3^2 + 2c_1c_2\phi_1\phi_2 + 2c_1c_3\phi_1\phi_3 + 2c_2c_3\phi_2\phi_3$

$\therefore \int_a^b f^2 dx = c_1^2\int_a^b \phi_1^2\,dx + c_2^2\int_a^b \phi_2^2\,dx + c_3^2\int_a^b \phi_3^2\,dx$

$+2c_1c_2\int_a^b \phi_1\phi_2 dx + 2c_1c_3\int_a^b \phi_1\phi_3 dx + 2c_2c_3\int_a^b \phi_2\phi_3\,dx$

using equation (1), (2) we get

$\int_a^b \left[f(x)\right]^2 dx = c_1^2 + c_2^2 + c_3^2$

**Q.5(c)** **Find inverse Laplace Transform of** **[8]**

**(i)** $\log\left(1 + \frac{a^2}{s^2}\right)$ **(ii)** $\frac{e^{-s}}{s^2 + s + 1}$

**Ans.:** (i) Let f(s) $= \log\left(1 + \frac{a^2}{s^2}\right) = \log\left(\frac{s^2 + a^2}{s^2}\right)$

$f(s) = \log(s^2 + a^2) - \log s^2$

$\Rightarrow f'(s) = \frac{2s}{s^2 + a^2} - \frac{2}{s}$

taking $L^{-1}$ we get

$L^{-1}\left\{f'(s)\right\} = 2\cos at - 2$

$\Rightarrow -t\,f(t) = 2\cos at - 2$

$\Rightarrow f(t) = \frac{2(1 - \cos at)}{t}$

(ii) Let $f(s) = \dfrac{1}{s^2+s+1} = \dfrac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$

$\therefore \quad L^{-1}\{F(s)\} = L^{-1}\left\{\dfrac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\}$

$\Rightarrow \quad F(t) = e^{-\frac{1}{2}t} L^{-1}\left\{\dfrac{1}{s^2+\frac{3}{4}}\right\} = \dfrac{e^{-\frac{1}{2}t}}{\frac{\sqrt{3}}{2}} \sin\left(\dfrac{\sqrt{3}}{2}t\right)$

we know that,

$L^{-1}\{e^{-as} f(s)\} = f(t-a)\, H(t-a)$

$\therefore \quad L^{-1}\left\{\dfrac{e^{-s}}{s^2+s+1}\right\} = \dfrac{2}{\sqrt{3}} e^{-\frac{1}{2}(t-1)} \sin\left[\dfrac{\sqrt{3}}{2}(t-1)\right]\cdot H(t-1)$

**Q.6(a) Obtain complex form of fourier series for F(x) = $e^{\alpha x}$, in $(-\pi, \pi)$ where a is not an integer.** **[6]**

**Ans.:** Complex form of fourier series is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad \ldots(1)$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax} e^{-inx}\, dx = \frac{1}{2\pi}\left\{\frac{e^{ax-inx}}{a-in}\right\}_{-\pi}^{\pi} = \frac{1}{2\pi}\frac{(a+in)}{(a+in)(a-in)}\left\{e^{a\pi}e^{-in\pi} - e^{-a\pi}e^{in\pi}\right\}$$

$$= \frac{1}{2\pi}\frac{(a+in)}{(a^2+n^2)}(-1)^n\left(e^{a\pi} - e^{-a\pi}\right)$$

$$c_n = \frac{(a+in)}{\pi(a^2+n^2)}(-1)^n \sinh(a\pi)$$

using this in (1)

$$e^{ax} = \sum_{n=-\infty}^{\infty} \frac{(a+in)}{\pi(a^2+n^2)}(-1)^n \sinh(a\pi)\, e^{inx}$$

**Q.6(b) Prove that $\bar{f}$ = $(ye^{xy}\cos z)i + (xe^{xy}\cos z)j + (-e^{xy}\sin z)k$ is irotational.** **[6]**
**Also find scalar potential $\phi$ and work done in moving particle from (0, 0, 0) to $(-1, 2, \pi)$**

**Ans.:**

$$\nabla\times\bar{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^{xy}\cos z & xe^{xy}\cos z & -e^{xy}\sin z \end{vmatrix}$$

$$= \left(-xe^{xy}\sin z + xe^{xy}\sin z\right)i - \left(-ye^{xy}\sin z + ye^{xy}\sin z\right)j$$
$$+\left(e^{xy}\cos z + xy\,e^{xy}\cos z - e^{xy}\cos z - xy\,e^{xy}\cos z\right)k$$

$= 0i + 0j + 0k$

$\Rightarrow \quad \bar{f}$ is irrotational.

Scalar potential $\phi = \int (ye^{xy}\cos z)dx + xe^{xy}\cos z\,dy - e^{xy}\sin z\,dz + c$

$\phi = e^{xy}\cos z + c$

work done $= \int_{(0,0,0)}^{(-1,2,\pi)} d\phi = \{e^{xy}\cos z\}_{(0,0,0)}^{(-1,2,\pi)}$

work done $= -e^{-2} - 1$

**Q.6(c) Find imaginary part of analytic function whose real part is $e^{2x}(x\cos 2y - y\sin 2y)$, Also verify that v is harmonic function.** **[8]**

**Ans.:** Given $u = e^{2x}(x\cos 2y - y\sin 2y)$

$f(z) = u + iv$ analytic $\Rightarrow u_x = v_y,\ v_x = -u_y \quad \} \rightarrow (1)$

$dv = v_x dx + v_y\,dy = -uy\,dx + u_x dy \quad [\because (1)]$

$dv = -e^{2x}(-2x\sin 2y - \sin 2y - 2y\cos 2y)\,dx$
$+\{e^{2x}(\cos 2y) + 2e^{2x}(x\cos 2y - y\sin 2y)dy\}$

which is exact.

by integration,

$v = \int_{y=\text{cont}} e^{2x}(2x\sin 2y + \sin 2y + 2y\cos 2y)\,dx + \int_{\text{free from } x} 0dy + c$

$= (2x\sin 2y + \sin 2y + 2y\cos 2y)\left(\frac{e^{2x}}{2}\right) - (2\sin 2y)\left(\frac{e^{2x}}{4}\right) + c$

$= \frac{e^{2x}}{2}\{2x\sin 2y + \sin 2y + 2y\cos 2y - \sin 2y\}$

$v = e^{2x}(x\sin 2y + y\cos 2y)$

$v_x = 2e^{2x}(x\sin 2y + y\cos 2y) + e^{2x}\sin 2y$

$v_{xx} = 4e^{2x}(x\sin 2y + y\cos 2y) + 2e^{2x}\sin 2y + 2e^{2x}\sin 2y$

$v_{xx} = 4xe^{2x}\sin 2y + 4y\,e^{2x}\cos 2y + 4\,e^{2x}\sin 2y$ ...(2)

$v_y = e^{2x}(2x\cos 2y + \cos 2y - 2y\sin 2y)$

$v_{yy} = e^{2x}(-4x\sin 2y - 2\sin 2y - 2\sin 2y - 4y\cos 2y)$

$v_{yy} = -4xe^{2x}\sin 2y - 4e^{2x}\sin 2y - 4y\,e^{2x}\cos 2y$ ...(3)

adding (2), (3) $V_{xx} + V_{yy} = 0$

$\Rightarrow$ v is harmonic.

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