

S.E. Sem. III [EXTC]
Applied Mathematics - III

Time : 3 Hrs.]

Prelim Paper Solution

[Marks : 80

Q.1(a) Find Laplace transform of $te^{3t} \cos t$. [5]

Ans.: $L\{\cos t\} = \frac{s}{s^2+1}$, $L\{t \cos t\} = (-1) \frac{d}{ds} \frac{s}{s^2+1}$

$$\therefore L\{t \cos t\} = - \left\{ \frac{s^2+1-s(2s)}{(s^2+1)^2} \right\} = \frac{s^2-1}{(s^2+1)^2}$$

by F.S.T.

$$L\{t e^{3t} \cos t\} = \frac{(s-3)^2-1}{[(s-3)^2+1]^2}$$

Q.1(b) Prove that : [5]

$\vec{f} = (x + 2y + az)\mathbf{i} + (bx - 3y - z)\mathbf{j} + (4x + cy + 2z)\mathbf{k}$
is solenoidal and determine the constants a, b, c if \vec{f} is irrotational.

Ans.: $\nabla \cdot \vec{f} = 1 - 3 + 2 = 0 \Rightarrow \vec{f}$ is solenoidal.

given $\nabla \times \vec{f} = 0$

$$\Rightarrow \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = 0$$

$$(c+1)\mathbf{i} - (4-a)\mathbf{j} + (b-2)\mathbf{k} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$$

$$\Rightarrow c = -1, a = 4, b = 2$$

Q.1(c) Show that $f(z) = \sinh z$ is analytic. Hence find its derivative. [5]

Ans.: $f(z) = \sinh z = \sinh(x + iy)$

$$= \sinh x \cosh iy + \cosh x \sinh iy$$

$$u + iv = \sinh x \cos y + i \cosh x \sin y$$

$$\Rightarrow u = \sinh x \cos y$$

$$u_x = \cosh x \cos y \quad \dots(1)$$

$$u_y = \sinh x \sin y \quad \dots(2)$$

$$v = \cosh x \sin y$$

$$v_x = \sinh x \sin y \quad \dots(3)$$

$$v_y = \cosh x \cos y \quad \dots(4)$$

From (1), (4) and (2), (3) $u_x = v_y$, $v_x = -u_y$

$\therefore f(z) = \sinh z$ is analytic.

$$\text{Now } F'(z) = u_x + iv_x \Big|_{x=z, y=0}$$

$$= \cosh x \cos y + i \sinh x \sin y \Big|_{x=z, y=0}$$

$$f'(z) = \cosh z$$

Q.1(d) Prove that $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cdot \cos x$ [5]

Ans.: we know that $\frac{d}{dx} [x^n J_n(x)] = x^n J_{n-1}(x)$

$$\Rightarrow \frac{d}{dx} x^{\frac{1}{2}} J_{\frac{1}{2}}(x) = x^{\frac{1}{2}} J_{-\frac{1}{2}}$$

$$\Rightarrow \frac{d}{dx} \left[x^{\frac{1}{2}} \sqrt{\frac{2}{\pi x}} \sin x \right] = x^{\frac{1}{2}} J_{-\frac{1}{2}}$$

$$\sqrt{\frac{2}{\pi}} \frac{d}{dx} \sin x = x^{\frac{1}{2}} J_{-\frac{1}{2}} \Rightarrow \sqrt{\frac{2}{\pi}} \cos x = x^{\frac{1}{2}} J_{-\frac{1}{2}}$$

$$\Rightarrow J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

Q.2(a) Show that the function $\omega = \frac{4}{z}$ transforms the straight line $x = c$ in the z -plane into circle in w -plane. Find its centre and radius. [6]

Ans.: $x = c$... (1)

Given $\omega = \frac{4}{z} \Rightarrow z = \frac{4}{\omega} \Rightarrow x + iy = \frac{4}{u + iv}$

$$\therefore x + iy = \frac{4u}{u^2 + v^2} - \frac{i4v}{u^2 + v^2} \Rightarrow x = \frac{4u}{u^2 + v^2}$$

using this in (1) $\frac{4u}{u^2 + v^2} = c \Rightarrow u^2 + v^2 - \frac{4u}{c} = 0$

centre = $\left(\frac{z}{c}, 0\right)$, rad = $\frac{2}{c}$

Q.2(b) Show that $\int_0^\infty e^{-t} \int_0^t \frac{\sin u}{u} du dt = \frac{\pi}{4}$ [6]

Ans.: $\int_0^\infty e^{-t} \int_0^t \frac{\sin u}{u} du dt = L \left\{ \int_0^t \frac{\sin u}{u} du \right\} \Big|_{s=1}$

Now

$$L \left\{ \int_0^t \frac{\sin u}{u} du \right\} = \frac{1}{s} L \left\{ \frac{\sin u}{u} \right\} = \frac{1}{s} \int_s^\infty \frac{1}{s^2 + 1} ds$$

$$= \frac{1}{s} \left\{ \tan^{-1} s \right\}_{s=s}^{s=\infty} = \frac{1}{s} \left[\frac{\pi}{2} - \tan^{-1} s \right]$$

\therefore at $s = 1$,

$$= \frac{1}{1} \left[\frac{\pi}{2} - \tan^{-1} 1 \right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Q.2(c) Obtain fourier series for $f(x) = \begin{cases} 1 + \frac{2x}{\pi} & -\pi \leq x \leq 0 \\ 1 - \frac{2x}{\pi} & 0 \leq x \leq \pi \end{cases}$ [8]

Hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

Ans.: $f(x)$ is even function. $\Rightarrow b_n = 0$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) \quad \dots(1)$$

$$a_0 = \frac{1}{2\pi} 2 \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) dx = \frac{1}{\pi} \left[x - \frac{x^2}{\pi} \right]_0^{\pi} \Rightarrow a_0 = 0$$

$$a_n = \frac{1}{\pi} 2 \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) \cos(nx) dx = \frac{2}{\pi} \left\{ \left(1 - \frac{2x}{\pi}\right) \left(\frac{\sin(nx)}{n}\right) - \left(-\frac{2}{\pi}\right) \left(\frac{-\cos(nx)}{n^2}\right) \right\}_0^{\pi}$$

$$= \frac{2}{\pi} \left\{ \left[0 - \frac{2}{\pi n^2} (-1)^n \right] - \left[0 - \frac{2}{\pi n^2} \right] \right\}$$

$$a_n = \frac{4}{n^2 \pi^2} [1 - (-1)^n]$$

using this in (1)

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n^2 \pi^2} [1 - (-1)^n] \cos(nx)$$

for deduction put $x = 0$ and note that $f(0) = 1$

$$1 = \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} [1 - (-1)^n]$$

$$1 = \frac{4}{\pi^2} \left[\frac{2}{1^2} + 0 + \frac{2}{3^2} + 0 + \frac{2}{5^2} + \dots \right] \Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Q.3(a) Evaluate by Green's Theorem $\int_C (e^{-x} \sin y \, dx + e^{-x} \cos y \, dy)$ [6]

where C is rectangle with vertices $(0, 0)$, $(\pi, 0)$, $(\pi, \frac{\pi}{2})$, $(0, \frac{\pi}{2})$

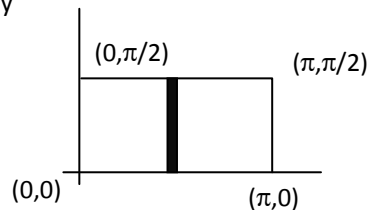
Ans.: $\int Pdx + Qdy = \iint (Q_x - P_y) dx dy$, $P = e^{-x} \sin y$, $Q = e^{-x} \cos y$

$$= \iint (-e^{-x} \cos y - e^{-x} \cos y) dx dy$$

$$= -2 \int_0^{\pi} \int_0^{\frac{\pi}{2}} e^{-x} \cos y \, dy \, dx$$

$$= -2 \int_0^{\pi} e^{-x} (\sin y)_0^{\frac{\pi}{2}} dx$$

$$= -2 \left(\frac{e^{-x}}{-1} \right)_0^{\pi} = 2(e^{-\pi} - 1)$$



Q.3(b) Prove that $J'_2(x) = \left(1 - \frac{4}{x^2}\right)J_1(x) + \frac{2}{x}J_0(x)$ [6]

Ans.: We know that

$$xJ'_n = -nJ_n + xJ_{n-1}$$

put $n = 2$, $xJ'_2 = -2J_2 + xJ_1 \Rightarrow J'_2 = -\frac{2}{x}J_2 + J_1 \rightarrow (1)$

by recurrence formula, $J'_n = \frac{n}{x}J_n - J_{n+1}$

also $J'_n = \frac{-n}{x}J_n + J_{n-1}$

equating

$$\frac{n}{x}J_n - J_{n+1} = \frac{-n}{x}J_n + J_{n-1}$$

$$\Rightarrow J_{n+1} = \frac{2n}{x}J_n - J_{n-1}$$

put $n = 1$, $J_2 = \frac{2}{x}J_1 - J_0$ using this in (1)

$$J'_2 = -\frac{2}{x} \left[\frac{2}{x}J_1 - J_0 \right] + J_1$$

$$J'_2 = \left(\frac{-4}{x^2} + 1 \right) J_1 + \frac{2}{x} J_0$$

Q.3(c) Solve the differential equation $\frac{dy}{dx} + 2y + \int_0^t y dt = \sin t$ using Laplace [8]

transform give $y(0) = 1$

Ans.: $L\{y'(t)\} + 2L\{y(t)\} + L\left\{\int_0^t y(t) dt\right\} = L\{\sin t\}$

$$5y(s) - 1 + 2y(s) + \frac{1}{s}y(s) = \frac{1}{s^2 + 1}$$

$$\left(s + 2 + \frac{1}{s}\right)y(s) = \frac{1}{s^2 + 1} + 1 = \frac{s^2 + 2}{s^2 + 1}$$

$$\frac{(s^2 + 2s + 1)}{s}y(s) = \frac{s^2 + 2}{s^2 + 1} \Rightarrow y(s) = \frac{s(s^2 + 2)}{(s + 1)^2(s^2 + 1)}$$

$$y(s) = \frac{a}{s + 1} + \frac{b}{(s + 1)^2} + \frac{cs + d}{s^2 + 1}, a = 1, b = -\frac{3}{2}, c = 0, d = \frac{1}{2}$$

$$\therefore y(s) = \frac{1}{s + 1} - \frac{3}{2} \frac{1}{(s + 1)^2} + \frac{1}{2} \frac{1}{s^2 + 1}$$

taking inverse Laplace,

$$y(t) = e^{-t} - \frac{3}{2}te^{-t} + \frac{1}{2}\sin t$$

Q.4(a) Find orthogonal trajectory to the family of curves $e^{-x} \cos y + xy = \text{constant}$ in $X - Y$ plane. [6]

Ans.: If $f(z) = u + iv$ analytic then v is orthogonal to u

Let $u = e^{-x} \cos y + xy$

$$dv = v_x dx + v_y dy = -u_y dx + u_x dy \quad [\because u_x = v_y, v_x = -u_y]$$

$$dv = -(-e^{-x} \sin y + x) + (-e^{-x} \cos y + y) dy$$

which is exact D.E.

integrating

$$v = \int_{y=\text{cont}} (e^{-x} \sin y + x) dx + \int_{\text{free from } x} (-e^{-x} \cos y + y) dy + c$$

$$v = -e^{-x} \sin y + \left(\frac{x^2}{2}\right) + \left(\frac{y^2}{2}\right) + c$$

Q.4(b) Show that $\cos x = 8\pi \sum_{m=1}^{\infty} \frac{m}{4m^2 - 1} \sin(2mx)$, if $0 < x < \pi$ [6]

Ans.: Half range sine series $f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \dots(1)$

$$b_n = \frac{2}{\pi} \int_0^{\pi} \cos x \sin(nx) dx = \frac{2}{\pi} \int_0^{\pi} \left[\frac{\sin(nx+x) + \sin(nx-x)}{2} \right] dx$$

$$= \frac{1}{\pi} \left[\frac{-\cos(nx+x)}{n+1} - \frac{\cos(nx-x)}{n-1} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left\{ \left[\frac{(-1)^n}{n+1} + \frac{(-1)^n}{n-1} \right] - \left[\frac{1}{n+1} - \frac{1}{n-1} \right] \right\}$$

$$b_n = \frac{1}{\pi} [1 + (-1)^n] \left[\frac{1}{n+1} + \frac{1}{n-1} \right] = \frac{2n}{\pi(n^2 - 1)} [1 + (-1)^n]$$

$n \neq 1$

$$\text{Now } b_1 = \frac{2}{\pi} \int_0^{\pi} \sin x \cos x dx = \frac{1}{\pi} \int_0^{\pi} \sin 2x dx$$

$$= \frac{1}{\pi} \left[-\frac{\cos 2x}{2} \right]_0^{\pi} = -\frac{1}{2\pi} [1 - 1] \Rightarrow b_1 = 0$$

$$\Rightarrow \cos x = \sum_{n=2}^{\infty} \frac{2n}{\pi(n^2 - 1)} [1 + (-1)^n] \sin(nx)$$

put $n = 2m$

$$\therefore \cos x = \sum_{2m=2}^{\infty} \frac{2(2m)}{\pi(4m^2 - 1)} [1 + (-1)^{2m}] \sin(2mx)$$

$$\therefore \cos x = \frac{8}{\pi} \sum_{m=1}^{\infty} \frac{m}{4m^2 - 1} \sin(2mx)$$

Q.4(c) Find Bilinear transformation which maps the points 1, i, -1 onto the points i, 0, -i. Hence find fixed points and image of $|z| < 1$. [8]

Ans.: Consider B. l. $\omega = \frac{az+b}{cz+d}$... (1)

given $z = 1, \omega = i$ (1)

$$\Rightarrow i = \frac{a+b}{c+d} \Rightarrow ic + id = a + b \quad \dots(2)$$

given $z = i, \omega = 0$ (1)

$$\Rightarrow 0 = \frac{ia+b}{ic+d} \Rightarrow b = -ia \quad \dots(3)$$

given $z = -1, \omega = -i$ (1)

$$\Rightarrow -i = \frac{-a+b}{-c+d} \Rightarrow ic - id = -a + b \quad \dots(4)$$

$$(2) + (4) \Rightarrow 2ic = 2b \Rightarrow 2ic = -2ia \Rightarrow c = -a \quad \dots(5)$$

$$(2) - (4) \Rightarrow 2id = 2a \Rightarrow d = -ia \quad \dots(6)$$

using (3), (5), (6), in (1)

$$\omega = \frac{az - ia}{-az - ia} \Rightarrow \omega = \frac{i-z}{i+z} \quad \dots(7)$$

For fixed points $w = z$, (7) $\Rightarrow z = \frac{i-z}{i+z}$

$$z^2 + iz = i - z \Rightarrow z^2 + (i+1)z - i = 0$$

$$z = \frac{-i-1 \pm \sqrt{-1+1+2i+4i}}{2} = \frac{-i-1 \pm \sqrt{6i}}{2}$$

to find image of $|z| < 1$: (7) $\Rightarrow i\omega + \omega z = i - z \Rightarrow (1 + \omega)z = i - i\omega$

$$z = \frac{i - i(u + iv)}{1 + u + iv}$$

$$\therefore |z| < 1 \Rightarrow \left| \frac{i - iu + v}{1 + u + iv} \right| < 1$$

$$|i - iu + v| = |1 + u + iv|$$

$$\sqrt{v^2 + (1-u)^2} < \sqrt{(1+u)^2 + v^2}$$

$$\Rightarrow v^2 + 1 - 2u + u^2 < 1 + 2u + u^2 + v^2 \Rightarrow 0 < 4u$$

$$\Rightarrow 0 < u$$

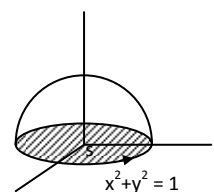
Q.5(a) Use Stoke's Theorem to evaluate $\int_c \bar{f} \cdot d\bar{r}$ [6]

Where $\bar{f} = yi + zj + xk$ and c is boundary of the surface $x^2 + y^2 = 1 - z, z > 0$.

Ans.:
$$\nabla \times \bar{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$\nabla \times \bar{f} = -i - j - k, \quad n'' = k = 0i + 0j + k$$

$$ds = dx dy$$



$$\begin{aligned} \therefore \int_c \bar{f} \cdot d\bar{r} &= \iint (\nabla \times \bar{f}) \cdot \bar{n}'' ds = \iint -dx dy = -\text{Area of circle} \\ &= -\pi(1)^2 = -\pi \end{aligned}$$

Q.5(b) If $f(x) = c_1 \phi_1(x) + c_2 \phi_2(x) + c_3 \phi_3(x)$, where c_1, c_2, c_3 are constant and ϕ_1, ϕ_2, ϕ_3 are orthonormal functions, on the set (a, b) . [6]

Show that $\int_a^b [f(x)]^2 dx = c_1^2 + c_2^2 + c_3^2$

Ans.: Given ϕ_1, ϕ_2, ϕ_3 are orthonormal on (a, b)

$$\Rightarrow \int_a^b \phi_1^2 dx = \int_a^b \phi_2^2 dx = \int_a^b \phi_3^2 dx = 1 \quad \dots(1)$$

$$\int_a^b \phi_1 \phi_2 dx = \int_a^b \phi_1 \phi_3 dx = \int_a^b \phi_2 \phi_3 dx = 0 \quad \dots(2)$$

Now

$$\begin{aligned} f &= c_1 \phi_1 + c_2 \phi_2 + c_3 \phi_3 && \text{squaring} \\ f^2 &= c_1^2 \phi_1^2 + c_2^2 \phi_2^2 + c_3^2 \phi_3^2 + 2c_1 c_2 \phi_1 \phi_2 + 2c_1 c_3 \phi_1 \phi_3 + 2c_2 c_3 \phi_2 \phi_3 \\ \therefore \int_a^b f^2 dx &= c_1^2 \int_a^b \phi_1^2 dx + c_2^2 \int_a^b \phi_2^2 dx + c_3^2 \int_a^b \phi_3^2 dx \\ &\quad + 2c_1 c_2 \int_a^b \phi_1 \phi_2 dx + 2c_1 c_3 \int_a^b \phi_1 \phi_3 dx + 2c_2 c_3 \int_a^b \phi_2 \phi_3 dx \end{aligned}$$

using equation (1), (2) we get

$$\int_a^b [f(x)]^2 dx = c_1^2 + c_2^2 + c_3^2$$

Q.5(c) Find inverse Laplace Transform of

[8]

(i) $\log\left(1 + \frac{a^2}{s^2}\right)$ (ii) $\frac{e^{-s}}{s^2 + s + 1}$

Ans.: (i) Let $f(s) = \log\left(1 + \frac{a^2}{s^2}\right) = \log\left(\frac{s^2 + a^2}{s^2}\right)$

$$f(s) = \log(s^2 + a^2) - \log s^2$$

$$\Rightarrow f'(s) = \frac{2s}{s^2 + a^2} - \frac{2}{s}$$

taking L^{-1} we get

$$L^{-1}\{f'(s)\} = 2 \cos at - 2$$

$$\Rightarrow -t f(t) = 2 \cos at - 2$$

$$\Rightarrow f(t) = \frac{2(1 - \cos at)}{t}$$

$$(ii) \text{ Let } f(s) = \frac{1}{s^2 + s + 1} = \frac{1}{\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$\therefore L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}}\right\}$$

$$\Rightarrow F(t) = e^{-\frac{1}{2}t} L^{-1}\left\{\frac{1}{s^2 + \frac{3}{4}}\right\} = \frac{e^{-\frac{1}{2}t}}{\frac{\sqrt{3}}{2}} \sin\left(\frac{\sqrt{3}}{2}t\right)$$

we know that,

$$L^{-1}\{e^{-as} f(s)\} = f(t-a) H(t-a)$$

$$\therefore L^{-1}\left\{\frac{e^{-s}}{s^2 + s + 1}\right\} = \frac{2}{\sqrt{3}} e^{-\frac{1}{2}(t-1)} \sin\left[\frac{\sqrt{3}}{2}(t-1)\right] \cdot H(t-1)$$

Q.6(a) Obtain complex form of fourier series for $F(x) = e^{\alpha x}$, in $(-\pi, \pi)$ where a is not an integer. [6]

Ans.: Complex form of fourier series is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \dots(1)$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ax} e^{-inx} dx = \frac{1}{2\pi} \left\{ \frac{e^{ax-inx}}{a-in} \right\}_{-\pi}^{\pi} = \frac{1}{2\pi} \frac{(a+in)}{(a-in)} \{e^{a\pi} e^{-in\pi} - e^{-a\pi} e^{in\pi}\}$$

$$= \frac{1}{2\pi} \frac{(a+in)}{(a^2+n^2)} (-1)^n (e^{a\pi} - e^{-a\pi})$$

$$c_n = \frac{(a+in)}{\pi(a^2+n^2)} (-1)^n \sinh(a\pi)$$

using this in (1)

$$e^{ax} = \sum_{n=-\infty}^{\infty} \frac{(a+in)}{\pi(a^2+n^2)} (-1)^n \sinh(a\pi) e^{inx}$$

Q.6(b) Prove that $\vec{f} = (ye^{xy} \cos z)\mathbf{i} + (xe^{xy} \cos z)\mathbf{j} + (-e^{xy} \sin z)\mathbf{k}$ is irrotational. [6]
Also find scalar potential ϕ and work done in moving particle from $(0, 0, 0)$ to $(-1, 2, \pi)$

Ans.:
$$\nabla \times \vec{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^{xy} \cos z & xe^{xy} \cos z & -e^{xy} \sin z \end{vmatrix}$$

$$= (-xe^{xy} \sin z + xe^{xy} \sin z)\mathbf{i} - (-ye^{xy} \sin z + ye^{xy} \sin z)\mathbf{j}$$

$$+ (e^{xy} \cos z + xy e^{xy} \cos z - e^{xy} \cos z - xy e^{xy} \cos z)\mathbf{k}$$

$$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$$

$$\Rightarrow \vec{f} \text{ is irrotational.}$$

$$\text{Scalar potential } \phi = \int (ye^{xy} \cos z) dx + xe^{xy} \cos z dy - e^{xy} \sin z dz + c$$

$$\phi = e^{xy} \cos z + c$$

$$\text{work done} = \int_{(0,0,0)}^{(-1,2,\pi)} d\phi = \left\{ e^{xy} \cos z \right\}_{(0,0,0)}^{(-1,2,\pi)}$$

$$\text{work done} = -e^{-2} - 1$$

**Q.6(c) Find imaginary part of analytic function whose real part is $e^{2x}(x \cos 2y - y \sin 2y)$, [8]
Also verify that v is harmonic function.**

Ans.: Given $u = e^{2x}(x \cos 2y - y \sin 2y)$

$$f(z) = u + iv \text{ analytic} \Rightarrow u_x = v_y, v_x = -u_y \quad \} \rightarrow (1)$$

$$dv = v_x dx + v_y dy = -u_y dx + u_x dy \quad [\because (1)]$$

$$dv = -e^{2x}(-2x \sin 2y - \sin 2y - 2y \cos 2y) dx + \{e^{2x}(\cos 2y) + 2e^{2x}(x \cos 2y - y \sin 2y)dy\}$$

which is exact.

by integration,

$$v = \int_{y=\text{cont}} e^{2x}(2x \sin 2y + \sin 2y + 2y \cos 2y) dx + \int_{\text{free from } x} 0 dy + c$$

$$= (2x \sin 2y + \sin 2y + 2y \cos 2y) \left(\frac{e^{2x}}{2} \right) - (2 \sin 2y) \left(\frac{e^{2x}}{4} \right) + c$$

$$= \frac{e^{2x}}{2} \{2x \sin 2y + \sin 2y + 2y \cos 2y - \sin 2y\}$$

$$v = e^{2x}(x \sin 2y + y \cos 2y)$$

$$v_x = 2e^{2x}(x \sin 2y + y \cos 2y) + e^{2x} \sin 2y$$

$$v_{xx} = 4e^{2x}(x \sin 2y + y \cos 2y) + 2e^{2x} \sin 2y + 2e^{2x} \sin 2y$$

$$v_{xx} = 4xe^{2x} \sin 2y + 4ye^{2x} \cos 2y + 4e^{2x} \sin 2y \quad \dots(2)$$

$$v_y = e^{2x}(2x \cos 2y + \cos 2y - 2y \sin 2y)$$

$$v_{yy} = e^{2x}(-4x \sin 2y - 2 \sin 2y - 2 \sin 2y - 4y \cos 2y)$$

$$v_{yy} = -4xe^{2x} \sin 2y - 4e^{2x} \sin 2y - 4ye^{2x} \cos 2y \quad \dots(3)$$

$$\text{adding (2), (3) } v_{xx} + v_{yy} = 0$$

$\Rightarrow v$ is harmonic.

