

Q.1(a) Find Laplace transform of $t e^{3t} \cos t$. [5]

Ans.: $L\{\cos t\} = \frac{s}{s^2+1}$, $L\{t \cos t\} = (-1) \frac{d}{ds} \frac{s}{s^2+1}$

$$\therefore L\{t \cos t\} = - \left\{ \frac{s^2+1-s(2s)}{(s^2+1)^2} \right\} = \frac{s^2-1}{(s^2+1)^2}$$

by F.S.T.

$$L\{t e^{3t} \cos t\} = \frac{(s-3)^2-1}{[(s-3)^2+1]^2}$$

Q.1(b) Evaluate $\oint_c \frac{z-1}{z^2+2z+5} dz$, where c is $|z+1+i|=2$. [5]

Ans.: Let $I = \int_c \frac{z-1}{z^2+2z+5} dz$ $c: |z+1+i|=2$

for poles, $z^2+2z+5=0$

$$\Rightarrow z = -1+2i, \quad z = -1-2i$$

$$|-1+2i+1+i| = |0+3i| = 3 > 2 \quad \text{outside}$$

$$|-1-2i+1+i| = |0-i| = 1 < 2 \quad \Rightarrow \text{pole } z = -1-2i \text{ is inside}$$

$$\begin{aligned} \text{at } z = -1-2i \quad \text{Residue} &= \frac{z-1}{2z+2} \Big|_{z=-1-2i} = \frac{-1-2i-1}{-2-4i+2} \\ &= \frac{-2-2i}{-4i} = \frac{1+i}{2i} \end{aligned}$$

$$\text{by Residue Theorem } \int_c \frac{z-1}{z^2+2z+5} = 2\pi i \left(\frac{1+i}{2i} \right) = \pi(1+i)$$

Q.1(c) Show that $f(z) = \sinh z$ is analytic. Hence find its derivative. [5]

Ans.: $f(z) = \sinh z = \sinh(x+iy) = \sinh x \cosh iy + \cosh x \sinh iy$
 $u+iv = \sinh x \cos y + i \cosh x \sin y$

$$\Rightarrow u = \sinh x \cos y$$

$$u_x = \cosh x \cos y \quad \dots(1)$$

$$u_y = \sinh x \sin y \quad \dots(2)$$

$$v = \cosh x \sin y$$

$$v_x = \sinh x \sin y \quad \dots(3)$$

$$v_y = \cosh x \cos y \quad \dots(4)$$

From (1), (4) and (2), (3) $u_x = v_y$, $v_x = -u_y$

$\therefore f(z) = \sinh z$ is analytic.

$$\text{Now } f'(z) = u_x + iv_x \Big|_{x=z, y=0} = \cosh x \cos y + i \sinh x \sin y \Big|_{x=z, y=0}$$

$$f'(z) = \cosh z$$

Q.1(d) Compute spearman’s rank correlation for the data : [5]

X :	18	20	34	52	12
Y :	39	23	35	18	46

Ans.:

X	Y	R _x	R _y	d = R _x – R _y	d ²
18	39	4	2	2	4
20	23	3	4	-1	1
34	35	2	3	-1	1
52	18	1	5	-4	16
12	46	5	1	4	16

$$\sum d^2 = 38$$

$$R = 1 - \frac{6 \sum d^2}{n(n^2 - 1)} = 1 - \frac{6(38)}{5(24)} \Rightarrow R = -0.9$$

Q.2(a) Show that the function $\omega = \frac{4}{z}$ transforms the straight line $x = c$ in the z -plane into circle in w -plane. Find its centre and radius. [6]

Ans.: $x = c$... (1)

Given $\omega = \frac{4}{z} \Rightarrow z = \frac{4}{\omega} \Rightarrow x + iy = \frac{4}{u + iv}$

$$\therefore x + iy = \frac{4u}{u^2 + v^2} - \frac{i4v}{u^2 + v^2} \Rightarrow x = \frac{4u}{u^2 + v^2}$$

using this in (1) $\frac{4u}{u^2 + v^2} = c \Rightarrow u^2 + v^2 - \frac{4u}{c} = 0$

$$\text{centre} = \left(\frac{z}{c}, 0 \right), \text{rad} = \frac{2}{c}$$

Q.2(b) Show that $\int_0^\infty e^{-t} \int_0^t \frac{\sin u}{u} du dt = \frac{\pi}{4}$ [6]

Ans.: $\int_0^\infty e^{-t} \int_0^t \frac{\sin u}{u} du dt = L \left\{ \int_0^t \frac{\sin u}{u} du \right\} \Big|_{s=1}$

Now

$$\begin{aligned} L \left\{ \int_0^t \frac{\sin u}{u} du \right\} &= \frac{1}{s} L \left\{ \frac{\sin u}{u} \right\} = \frac{1}{s} \int_s^\infty \frac{1}{s^2 + 1} ds \\ &= \frac{1}{s} \left\{ \tan^{-1} s \right\}_{s=s}^{s=\infty} = \frac{1}{s} \left[\frac{\pi}{2} - \tan^{-1} s \right] \end{aligned}$$

\therefore at $s = 1,$

$$= \frac{1}{1} \left[\frac{\pi}{2} - \tan^{-1} 1 \right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Q.2(c) Obtain fourier series for $f(x) = \begin{cases} 1 + \frac{2x}{\pi} & -\pi \leq x \leq 0 \\ 1 - \frac{2x}{\pi} & 0 \leq x \leq \pi \end{cases}$ [8]

Hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

Ans.: $f(x)$ is even function. $\Rightarrow b_n = 0$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) \quad \dots(1)$$

$$a_0 = \frac{1}{2\pi} 2 \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) dx = \frac{1}{\pi} \left[x - \frac{x^2}{\pi} \right]_0^{\pi} \Rightarrow a_0 = 0$$

$$a_n = \frac{1}{\pi} 2 \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) \cos(nx) dx = \frac{2}{\pi} \left\{ \left(1 - \frac{2x}{\pi}\right) \left(\frac{\sin(nx)}{n}\right) - \left(-\frac{2}{\pi}\right) \left(\frac{-\cos(nx)}{n^2}\right) \right\}_0^{\pi}$$

$$= \frac{2}{\pi} \left\{ \left[0 - \frac{2}{\pi n^2} (-1)^n\right] - \left[0 - \frac{2}{\pi n^2}\right] \right\}$$

$$a_n = \frac{4}{n^2 \pi^2} [1 - (-1)^n]$$

using this in (1)

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n^2 \pi^2} [1 - (-1)^n] \cos(nx)$$

for deduction put $x = 0$ and note that $f(0) = 1$

$$1 = \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} [1 - (-1)^n]$$

$$1 = \frac{4}{\pi^2} \left[\frac{2}{1^2} + 0 + \frac{2}{3^2} + 0 + \frac{2}{5^2} + \dots \right] \Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Q.3(a) Evaluate $\oint_c \frac{e^{kz}}{z} dz$, where $c: |z| = 1$. [6]

Hence deduce that $\int_0^{\pi} e^{k \sin \theta} \cos(k \sin \theta) d\theta = \pi$

Ans.: $I = \int_c \frac{e^{kz}}{z} dz$, $c: |z| = 1$

pole $z = 0$ is inside $|z| = 1$

$$\therefore \int_c \frac{e^{kz}}{z} dz = 2\pi i f(0), \quad \text{where } f(z) = e^{kz}$$

$$= 2\pi i \quad (1)$$

put $z = e^{i\theta} \Rightarrow dz = ie^{i\theta} d\theta$, limits 0 to 2π

$$\therefore (1) \Rightarrow \int_0^{2\pi} \frac{e^{ke^{i\theta}}}{e^{i\theta}} i e^{i\theta} d\theta = 2\pi i$$

$$i \int_0^{2\pi} e^{k(\cos \theta + i \sin \theta)} d\theta = 2\pi i$$

$$\int_0^{2\pi} e^{k \cos \theta} [\cos(k \sin \theta) + i \sin(k \sin \theta)] d\theta = 2\pi$$

$$\int_0^{2\pi} e^{k \cos \theta} \cos(k \sin \theta) d\theta + i \int_0^{2\pi} e^{k \cos \theta} \sin(k \sin \theta) d\theta = 2\pi + 0i$$

$$\Rightarrow \int_0^{2\pi} e^{k \cos \theta} \cos(k \sin \theta) d\theta = 2\pi$$

$$2 \int_0^{\pi} e^{k \cos \theta} \cos(k \sin \theta) d\theta = 2\pi$$

$$\Rightarrow \int_0^{\pi} e^{k \cos \theta} \cos(k \sin \theta) d\theta = \pi$$

Q.3(b) For the lines of regression $6y - 5x = 90$, $15x - 8y = 130$ and $\sigma_x^2 = 16$ [6]

Find (i) \bar{x} , \bar{y} (ii) r (iii) σ_y

Ans.: given $6y - 5x = 90$, $15x - 8y = 130$
solving we get $x = 30$, $y = 40$
i.e. $\bar{x} = 30$, $\bar{y} = 40$

$$y = \frac{5}{6}x + 15 \Rightarrow b_{yx} = \frac{5}{6}$$

$$x = \frac{8}{15}y + \frac{130}{15} \Rightarrow b_{xy} = \frac{8}{15}$$

$$\therefore r = \pm \sqrt{b_{xy} b_{yx}} = +\sqrt{\frac{5}{6} \cdot \frac{8}{15}} \Rightarrow r = 0.67$$

given $\sigma_x^2 = 16 \Rightarrow \sigma_x = 4$

$$b_{xy} = \frac{8}{15} \Rightarrow r \frac{\sigma_x}{\sigma_y} = \frac{8}{15} \Rightarrow \frac{2 \left(\frac{4}{\sigma_y} \right)}{3} = \frac{8}{15}$$

$$6\sigma_y = 5$$

Q.3(c) Solve the differential equation $\frac{dy}{dx} + 2y + \int_0^t y dt = \sin t$ using Laplace [8]

transform give $y(0) = 1$

Ans.: $L\{y'(t)\} + 2L\{y(t)\} + L\left\{\int_0^t y(t) dt\right\} = L\{\sin t\}$

$$5y(s) - 1 + 2y(s) + \frac{1}{s} y(s) = \frac{1}{s^2 + 1}$$

$$\left(s + 2 + \frac{1}{s}\right)y(s) = \frac{1}{s^2 + 1} + 1 = \frac{s^2 + 2}{s^2 + 1}$$

$$\frac{(s^2 + 2s + 1)}{s} y(s) = \frac{s^2 + 2}{s^2 + 1} \Rightarrow y(s) = \frac{s(s^2 + 2)}{(s+1)^2 (s^2 + 1)}$$

$$y(s) = \frac{a}{s+1} + \frac{b}{(s+1)^2} + \frac{cs+d}{s^2+1}, a=1, b=-\frac{3}{2}, c=0, d=\frac{1}{2}$$

$$\therefore y(s) = \frac{1}{s+1} - \frac{3}{2} \frac{1}{(s+1)^2} + \frac{1}{2} \frac{1}{s^2+1}$$

taking inverse Laplace,

$$y(t) = e^{-t} - \frac{3}{2} te^{-t} + \frac{1}{2} \sin t$$

Q.4(a) Find Laurent's series for $f(z) = \frac{2}{(z-1)(z-2)}$ indicating region of convergence. [6]

Ans.: $f(z) = \frac{2}{(z-1)(z-2)} = \frac{a}{z-1} + \frac{b}{z-2} = -\frac{2}{z-1} + \frac{2}{z-2} \dots(1)$

for Laurent's series ROC is (1) $|z| > 1, |z| > 2$ (ii) $1 < |z| < 2$

case (1): $|z| > 1, |z| > 2$

$$(1) \Rightarrow f(z) = \frac{-2}{z\left(1-\frac{1}{z}\right)} + \frac{2}{z\left(1-\frac{2}{z}\right)} = -\frac{2}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{2}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$f(z) = \frac{-2}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \dots\right] + \frac{2}{z}\left[1 + \frac{2}{z} + \frac{4}{z^2} + \dots\right]$$

case (2): $1 < |z| < 2$

$$(1) \Rightarrow f(z) = \frac{-2}{z\left(1-\frac{1}{z}\right)} - \frac{2}{2\left(1-\frac{z}{2}\right)} = -\frac{2}{z}\left(1-\frac{1}{z}\right)^{-1} - \left(1-\frac{z}{2}\right)^{-1}$$

$$f(z) = \frac{-2}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \dots\right] - \left[1 + \frac{z}{2} + \frac{z^2}{4} + \dots\right]$$

Q.4(b) Show that $\cos x = 8\pi \sum_{m=1}^{\infty} \frac{m}{4m^2-1} \sin(2mx)$, if $0 < x < \pi$ [6]

Ans.: Half range sine series $f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \dots(1)$

$$b_n = \frac{2}{\pi} \int_0^{\pi} \cos x \sin(nx) dx = \frac{2}{\pi} \int_0^{\pi} \left[\frac{\sin(nx+x)}{2} + \frac{\sin(nx-x)}{2} \right] dx$$

$$= \frac{1}{\pi} \left[\frac{-\cos(nx+x)}{n+1} - \frac{\cos(nx-x)}{n-1} \right]_0^{\pi} = \frac{1}{\pi} \left[\frac{(-1)^n}{n+1} + \frac{(-1)^n}{n-1} \right] - \left[-\frac{1}{n+1} - \frac{1}{n-1} \right]$$

$$b_n = \frac{1}{\pi} [1 + (-1)^n] \left[\frac{1}{n+1} + \frac{1}{n-1} \right] = \frac{2n}{\pi(n^2-1)} [1 + (-1)^n]$$

$n \neq 1$

$$\begin{aligned} \text{Now } b_1 &= \frac{2}{\pi} \int_0^{\pi} \sin x \cos x \, dx = \frac{1}{\pi} \int_0^{\pi} \sin 2x \, dx \\ &= \frac{1}{\pi} \left[-\frac{\cos 2x}{2} \right]_0^{\pi} = -\frac{1}{2\pi} [1-1] \Rightarrow b_1 = 0 \end{aligned}$$

$$\Rightarrow \cos x = \sum_{n=2}^{\infty} \frac{2n}{\pi(n^2-1)} [1+(-1)^n] \sin(nx)$$

put $n = 2m$

$$\therefore \cos x = \sum_{2m=2}^{\infty} \frac{2(2m)}{\pi(4m^2-1)} [1+(-1)^{2m}] \sin(2mx)$$

$$\therefore \cos x = \frac{8}{\pi} \sum_{m=1}^{\infty} \frac{m}{4m^2-1} \sin(2mx)$$

Q.4(c) Find Bilinear transformation which maps the points 1, i, -1 onto the points i, 0, -i. Hence find fixed points and image of $|z| < 1$. [8]

Ans.: Consider B. l. $\omega = \frac{az+b}{cz+d}$... (1)

given $z = 1, \omega = i$ (1)

$$\Rightarrow i = \frac{a+b}{c+d} \Rightarrow ic + id = a + b \quad \dots(2)$$

given $z = i, \omega = 0$ (1)

$$\Rightarrow 0 = \frac{ia+b}{ic+d} \Rightarrow b = -ia \quad \dots(3)$$

given $z = -1, \omega = -i$ (1)

$$\Rightarrow -i = \frac{-a+b}{-c+d} \Rightarrow ic - id = -a + b \quad \dots(4)$$

$$(2) + (4) \Rightarrow 2ic = 2b \Rightarrow 2ic = -2ia \Rightarrow c = -a \quad \dots(5)$$

$$(2) - (4) \Rightarrow 2id = 2a \Rightarrow d = -ia \quad \dots(6)$$

using (3), (5), (6), in (1)

$$\omega = \frac{az-ia}{-az-ia} \Rightarrow \omega = \frac{i-z}{i+z} \quad \dots(7)$$

For fixed points $w = z$, (7) $\Rightarrow z = \frac{i-z}{i+z}$

$$z^2 + iz = i - z \Rightarrow z^2 + (i+1)z - i = 0$$

$$z = \frac{-i-1 \pm \sqrt{-1+1+2i+4i}}{2} = \frac{-i-1 \pm \sqrt{6i}}{2}$$

to find image of $|z| < 1$: (7) $\Rightarrow i\omega + \omega z = i - z \Rightarrow (1 + \omega)z = i - i\omega$

$$z = \frac{i - i(u+iv)}{1+u+iv}$$

$$\therefore |z| < 1 \Rightarrow \left| \frac{i - iu + v}{1 + u + iv} \right| < 1$$

$$|i - iu + v| = |1 + u + iv|$$

$$\begin{aligned} \sqrt{v^2 + (1-u)^2} &< \sqrt{(1+u)^2 + v^2} \\ \Rightarrow v^2 + 1 - 2u + u^2 &< 1 + 2u + u^2 + v^2 \Rightarrow 0 < 4u \\ \Rightarrow 0 < u \end{aligned}$$

Q.5(a) Solve using Bender-Schmidt method $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$, subject to the conditions [6]

$u(0, t) = 0, u(1, t) = 0, u(x, 0) = \sin \pi x, 0 \leq x \leq 1$

Ans.: $a = 1$, take $h = 0.2 \Rightarrow x : 0, 0.2, 0.4, 0.6, 0.8, 1$

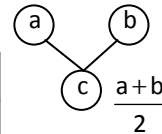
$$k = \frac{a}{2} h^2 = 0.02$$

$t : 0, 0.02, 0.04, 0.06, 0.08, 0.10$
 $x \rightarrow h = 0.2$

t ↓ k = 0.02	t \ x	0.0	0.2	0.4	0.6	0.8	1
	0.00	0	0.5878	0.9511	0.9511	0.5878	0
	0.02	0					0
	0.04	0					0
	0.06	0					0
	0.08	0					0
	0.10	0					0

using Bender-Schmidt formula $c = \frac{(a+b)}{2}$

t \ x	0	0.2	0.4	0.6	0.8	1
0	0	0.5878	0.9511	0.9511	0.5878	0
0.02	0	0.4756	0.7695	0.7695	0.4756	0
0.04	0	0.3848	0.6225	0.6225	0.3848	0
0.06	0	0.3113	0.5036	0.5036	0.3113	0
0.08	0	0.2518	0.4074	0.4074	0.2518	0
0.10	0	0.2037	0.3296	0.3296	0.2037	0



Q.5(b) Determine the solution of one-dimensional heat equation under the [6]

boundary conditions, $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

$u(0,t) = 0, u(\ell, t) = 0, u(x,0) = x, (0 < x < \ell), \ell$ being length of the rod.

Ans.: Solution of heat flow is given by

$$u = (c_1 \cos mx + c_2 \sin mx) e^{-m^2 c^2 t} \quad \dots(1)$$

given $u = 0$ when $x = 0$

$$(1) \Rightarrow 0 = c_1 e^{-m^2 c^2 t} \Rightarrow c_1 = 0$$

$$\therefore (1) \Rightarrow u = c_2 \sin mx e^{-m^2 c^2 t} \quad \dots(2)$$

given when $x = \ell, u = 0$

$$(2) \Rightarrow 0 = c \sin m \ell e^{-m^2 c^2 t} \Rightarrow \sin m \ell = 0 \Rightarrow m \ell = n\pi$$

$$\Rightarrow m = \frac{n\pi}{\ell}$$

$$(2) \Rightarrow u = c_2 \sin\left(\frac{n\pi x}{\ell}\right) e^{-\frac{n^2 \pi^2}{\ell^2} c^2 t} \quad \dots(3)$$

adding all above solutions for $n = 1, 2, \dots$ we get the general solution

$$u = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right) e^{-\frac{n^2 \pi^2 c^2 t}{\ell^2}} \quad \dots(3)$$

given when $t = 0, u = x$

$$(3) \Rightarrow x = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)$$

which is half range series in $(0, \ell)$

$$b_n = \frac{2}{\ell} \int_0^{\ell} x \sin\left(\frac{n\pi x}{\ell}\right) dx = \frac{2}{\ell} \left\{ (x) \left[\frac{-\cos\left(\frac{n\pi x}{\ell}\right)}{\frac{n\pi}{\ell}} \right] - (1) \left[\frac{-\sin\left(\frac{n\pi x}{\ell}\right)}{\frac{n^2 \pi^2}{\ell^2}} \right] \right\}_0^{\ell}$$

$$= \frac{2}{\ell} \left\{ \left[-\frac{\ell}{n\pi} (-1)^n - 0 \right] - [0 - 0] \right\} = -\frac{2\ell}{n\pi} (-1)^n$$

using this in (3) we get,

$$u = \sum_{n=1}^{\infty} \frac{-2\ell}{n\pi} (-1)^n \sin\left(\frac{n\pi x}{\ell}\right) e^{-\frac{n^2 \pi^2 c^2 t}{\ell^2}}$$

Q.5(c) Find inverse Laplace Transform of

[8]

(i) $\log\left(1 + \frac{a^2}{s^2}\right)$ (ii) $\frac{e^{-s}}{s^2 + s + 1}$

Ans.: (i) Let $f(s) = \log\left(1 + \frac{a^2}{s^2}\right) = \log\left(\frac{s^2 + a^2}{s^2}\right)$

$$f(s) = \log(s^2 + a^2) - \log s^2$$

$$\Rightarrow f'(s) = \frac{2s}{s^2 + a^2} - \frac{2}{s}$$

taking L^{-1} we get

$$L^{-1}\{f'(s)\} = 2 \cos at - 2$$

$$\Rightarrow -t f(t) = 2 \cos at - 2$$

$$\Rightarrow f(t) = \frac{2(1 - \cos at)}{t}$$

$$(ii) \text{ Let } f(s) = \frac{1}{s^2 + s + 1} = \frac{1}{\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$\therefore L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}}\right\}$$

$$\Rightarrow F(t) = e^{-\frac{1}{2}t} L^{-1}\left\{\frac{1}{s^2 + \frac{3}{4}}\right\} = \frac{e^{-\frac{1}{2}t}}{\frac{\sqrt{3}}{2}} \sin\left(\frac{\sqrt{3}}{2}t\right)$$

we know that,

$$L^{-1}\{e^{-as} f(s)\} = f(t-a) H(t-a)$$

$$\therefore L^{-1}\left\{\frac{e^{-s}}{s^2 + s + 1}\right\} = \frac{2}{\sqrt{3}} e^{-\frac{1}{2}(t-1)} \sin\left[\frac{\sqrt{3}}{2}(t-1)\right] \cdot H(t-1)$$

Q.6(a) Obtain complex form of fourier series for $F(x) = e^{\alpha x}$, in $(-\pi, \pi)$ where a is not an integer. [6]

Ans.: Complex form of fourier series is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \dots(1)$$

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ax} e^{-inx} dx = \frac{1}{2\pi} \left\{ \frac{e^{ax-inx}}{a-in} \right\}_{-\pi}^{\pi} = \frac{1}{2\pi} \frac{(a+in)}{(a-in)} \{e^{a\pi} e^{-in\pi} - e^{-a\pi} e^{in\pi}\} \\ &= \frac{1}{2\pi} \frac{(a+in)}{(a^2+n^2)} (-1)^n (e^{a\pi} - e^{-a\pi}) = \frac{(a+in)}{\pi(a^2+n^2)} (-1)^n \sinh(a\pi) \end{aligned}$$

using this in (1)

$$e^{ax} = \sum_{n=-\infty}^{\infty} \frac{(a+in)}{\pi(a^2+n^2)} (-1)^n \sinh(a\pi) e^{inx}$$

Q.6(b) Fit a curve $y = a \cdot b^x$ to the following data, using method of least squares. [6]

X :	2	3	4	5	6
Y :	144	172.8	207.4	248.8	298.5

Ans.: $y = a \cdot b^x$ (taking log) ... (1)

$$\log y = \log a + x \log b$$

$$\Rightarrow \sum \log y = n \ln a + \ln b \sum x \quad \dots(2)$$

$$\sum x \log y = \ln a \sum x + \ln b \sum x^2 \quad \dots(3)$$

$$\left. \begin{aligned} (2) &\Rightarrow 26.58 = 5A + 20B \\ (3) &\Rightarrow 108.51 = 20A + 90B \end{aligned} \right\} \text{ where } A = \ln a, B = \ln b$$

$$\text{Solving} \quad \ln a = 4.62$$

$$B = 0.179 \quad a = 101.49$$

$$\therefore (1) \Rightarrow \ln b = 0.179$$

$$y = (101.49) (1.196)^x \quad b = 1.196$$

Q.6(c) (i) Evaluate $\int_0^{2\pi} \frac{d\theta}{5 + 3 \sin \theta}$ using Residue Theorem. [4]

(ii) Evaluate $\int_{-\infty}^{\infty} \frac{dx}{x^2 + 1}$ using Residue Theorem. [4]

Ans.: (i) $I = \int_0^{2\pi} \frac{d\theta}{5 + 3 \sin \theta} = \text{put } z = e^{i\theta} \Rightarrow d\theta = \frac{dz}{iz}$

$$\sin \theta = \frac{z^2 - 1}{2iz}$$

$$\therefore I = \int_c \frac{\frac{dz}{iz}}{5 + 3 \left(\frac{z^2 - 1}{2i} \right)} = \int_c \frac{2}{3z^2 + 10iz - 3} dz$$

poles are $z = -\frac{i}{3}, z = -3i$

$z = -\frac{i}{3}$ lies inside $|z| = 1$

$$\text{at } z = -\frac{i}{3}, \text{ Res} = \frac{2}{6z + 10i} \Big|_{z = -\frac{i}{3}} = \frac{2}{-2i + 10i} = \frac{1}{4i} = \frac{1}{4i}$$

by Residue Theorem,

$$I = 2\pi i \left(\frac{1}{4i} \right) = \frac{\pi}{2}$$

(ii) Consider $\int_c \frac{dz}{z^2 + 1}$ where c is large semicircle as shown in figure.

$$\therefore \int_{-R}^R \frac{dz}{z^2 + 1} + \int_{C_1} \frac{dz}{z^2 + 1} = \int_c \frac{dz}{z^2 + 1} \quad \dots(1)$$

taking limit as $R \rightarrow \infty$

$$\text{along real axis } y = 0 \Rightarrow z = x \text{ and } \int_{C_1} \frac{dz}{z^2 + 1} = 0$$

$$\therefore (1) \Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} = \int_c \frac{dz}{z^2 + 1},$$

poles are $z = \pm i$ but $z = i$ lies inside

$$\text{at } z = i \text{ Res} = \frac{1}{2z} \Big|_{z=i} = \frac{1}{2i}$$

\therefore by residue theorem

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} = 2\pi i \left(\frac{1}{2i} \right) = \pi$$

